

Circling Infinity: String Art with Rational Numbers

Workshop worksheet

Properties of rational numbers

Theorem: Let n/q be an irreducible fraction ($n, q \in \mathbb{N}$).

- If $q = 2^\alpha 5^\beta$, and $\max(\alpha, \beta) = \mu$, then the decimal terminates after μ digits.
- If $q = 2^\alpha 5^\beta Q$, where $Q > 1$, $(Q, 10) = 1$, and $v = \text{ord}_Q 10$, where $(Q, 10)$ is the greatest common divisor of Q and 10 , and $\text{ord}_Q 10$ is the order of $10 \pmod{Q}$, i.e., the least positive integer k such that $10^k \equiv 1 \pmod{Q}$, then the decimal contains μ non-repeating and v repeating digits.

Theorem: Let n/q be an irreducible fraction ($n, q \in \mathbb{N}$). For prime denominators, q (coprime with 10), the number of distinct repetends of the decimals of irreducible fractions with denominator q is simply $(q - 1)/v$, where $v = \text{ord}_q 10$.

Midy's theorem: Given an irreducible proper fraction, p/q , where q is a prime (coprime with 10), and the corresponding repetend has even length $2m$, $n/q = 0.\overline{a_1 a_2 a_3 \dots a_n a_{n+1} \dots a_{2m}}$, then, for $1 \leq i \leq m$, $a_i + a_{m+i} = 9$.

Table 1: Cycles of pure repeating decimals of irreducible fractions with denominator q , q coprime with 10 (λ is the number of distinct cycles and v is the cycle length).

q	λ	v	Cycles
3	2	1	$\overline{3} \ \overline{6}$
7	1	6	$\overline{142857}$
9	6	1	$\overline{1} \ \overline{2} \ \overline{4} \ \overline{5} \ \overline{7} \ \overline{8}$
11	5	2	$\overline{09} \ \overline{18} \ \overline{27} \ \overline{36} \ \overline{45}$
13	2	6	$\overline{076923} \ \overline{153846}$
17	1	16	$\overline{0588235294117647}$
19	1	18	$\overline{052631578947368421}$
21	2	6	$\overline{047619} \ \overline{095238}$
23	1	22	$\overline{0434782608695652173913}$
27	6	3	$\overline{037} \ \overline{074} \ \overline{148} \ \overline{185} \ \overline{259} \ \overline{296}$
29	1	28	$\overline{0344827586206896551724137931}$
31	2	15	$\overline{032258064516129} \ \overline{096774193548387}$
33	10	2	$\overline{03} \ \overline{06} \ \overline{12} \ \overline{15} \ \overline{24} \ \overline{39} \ \overline{48} \ \overline{57} \ \overline{69} \ \overline{78}$
37	12	3	$\overline{027} \ \overline{054} \ \overline{081} \ \overline{135} \ \overline{162} \ \overline{189} \ \overline{243} \ \overline{297} \ \overline{378} \ \overline{459} \ \overline{486} \ \overline{567}$
39	4	6	$\overline{025641} \ \overline{051282} \ \overline{179487} \ \overline{358974}$
41	8	5	$\overline{02439} \ \overline{04878} \ \overline{07317} \ \overline{09756} \ \overline{12195} \ \overline{14634} \ \overline{26829} \ \overline{36585}$
43	2	21	$\overline{023255813953488372093} \ \overline{046511627906976744186}$
47	1	46	$\overline{0212765957446808510638297872340425531917893617}$
49	1	42	$\overline{020408163265306122448979591836734693877551}$
51	2	16	$\overline{0196078431372549} \ \overline{0392156862745098}$

Table 2: Paths of all decimals of fractions with denominator q , not included in Table 1.

q	Paths
1	$\overline{0} \equiv \overline{9}$
2	$\overline{50} \equiv \overline{49} \quad \overline{0} \equiv \overline{9}$
4	$\overline{250} \equiv \overline{249} \quad \overline{50} \equiv \overline{49} \quad \overline{750} \equiv \overline{749} \quad \overline{0} \equiv \overline{9}$
5	$\overline{20} \equiv \overline{19} \quad \overline{40} \equiv \overline{39} \quad \overline{60} \equiv \overline{69} \quad \overline{80} \equiv \overline{79} \quad \overline{0} \equiv \overline{9}$
6	$\overline{16} \quad \overline{3} \quad \overline{50} \equiv \overline{49} \quad \overline{6} \quad \overline{83} \quad \overline{0} \equiv \overline{9}$
8	$\overline{1250} \equiv \overline{1249} \quad \overline{250} \equiv \overline{249} \quad \overline{3750} \equiv \overline{3749} \quad \overline{50} \equiv \overline{49} \quad \overline{6250} \equiv \overline{6249} \quad \overline{750} \equiv \overline{749} \quad \overline{8750} \equiv \overline{8749} \quad \overline{0} \equiv \overline{9}$
10	$\overline{10} \equiv \overline{09} \quad \overline{20} \equiv \overline{19} \quad \overline{30} \equiv \overline{29} \quad \dots \quad \overline{80} \equiv \overline{79} \quad \overline{90} \equiv \overline{89} \quad \overline{0} \equiv \overline{9}$
12	$\overline{83} \quad \overline{16} \quad \overline{250} \equiv \overline{249} \quad \overline{3} \quad \overline{416} \quad \overline{50} \equiv \overline{49} \quad \dots \quad \overline{0} \equiv \overline{9}$
14	$\overline{0714285} \quad \overline{142857} \quad \overline{2142857} \quad \overline{285714} \quad \overline{3571428} \quad \overline{428571} \quad \overline{50} \equiv \overline{49} \quad \dots \quad \overline{0} \equiv \overline{9}$
15	$\overline{06} \quad \overline{13} \quad \overline{20} \equiv \overline{19} \quad \overline{26} \quad \overline{3} \quad \overline{40} \equiv \overline{39} \quad \overline{46} \quad \overline{53} \quad \overline{60} \equiv \overline{59} \quad \dots \quad \overline{0} \equiv \overline{9}$
16	$\overline{06250} \equiv \overline{06249} \quad \overline{1250} \equiv \overline{1249} \quad \overline{18750} \equiv \overline{18749} \quad \overline{250} \equiv \overline{249} \quad \dots \quad \overline{0} \equiv \overline{9}$
18	$\overline{05} \quad \overline{1} \quad \overline{16} \quad \overline{2} \quad \overline{27} \quad \overline{3} \quad \overline{38} \quad \overline{4} \quad \overline{50} \equiv \overline{49} \quad \overline{5} \quad \overline{61} \quad \overline{6} \quad \dots \quad \overline{0} \equiv \overline{9}$
20	$\overline{050} \equiv \overline{049} \quad \overline{10} \equiv \overline{09} \quad \overline{150} \equiv \overline{149} \quad \overline{20} \equiv \overline{19} \quad \overline{250} \equiv \overline{249} \quad \overline{30} \equiv \overline{29} \quad \dots \quad \overline{0} \equiv \overline{9}$

Template

