Topological Crochet

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Abstract

This paper discusses the theory and practice of topological crochet, a new way to make mathematical sculptures, which captures the topological essence of the final surface in each step. We also discuss the implication of this craft on sculpting in general.

Introduction

Traditional crafts like knitting, crochet, and weaving use string-like material to make surfaces. Most of the time, the focus is on the functional side of the final projects: knitting for wearables, crochet for blankets, and weaving for fabric and baskets. But knitting, crochet, and weaving have also been used to make abstract forms [11, 8, 6, 7]. In this paper, we discuss the art of topological crochet, in which the abstract forms are built up from a skinny version of the final model. Every step of the process preserves the shape's topology.

When it comes to making topologically nontrivial surfaces, such as the Seifert surface of a trefoil knot shown in Figure 1(a), the first idea that comes to most people is to chop the form into topologically trivial pieces that can be joined to create the surface. This is the mindset of a seamstress or differential geometer. The seamstresses' method can successfully create a model of the surface, but it has drawbacks. The seams are mathematically arbitrary and interrupt the flow of the final surface. Another method, usually used in knitting a pair of pants, is similar to 3D printing and involves building up several rounds while splitting or joining the rounds around a saddle point. This method is more elegant. However, it still distinguishes the top from the bottom. An abstract surface does not have a preferred direction most of the time.

Topological crochet is a method that abandons both arbitrary seams and the global direction. It is based on the concept of a *mapping cylinder* [2] from algebraic topology. Instead of joining pieces, we use foundation chains to build a ribbon graph of the final surface. Then, we build rounds along the boundaries of the ribbons to reach the desired size and shape of the surface. The flexibility of crochet makes both steps easy to implement and makes crochet an ideal way to make topologically interesting surfaces like those shown in Figure 1. This work was proceeded by those of Matthew Wright [13] and Moira Chas [1].

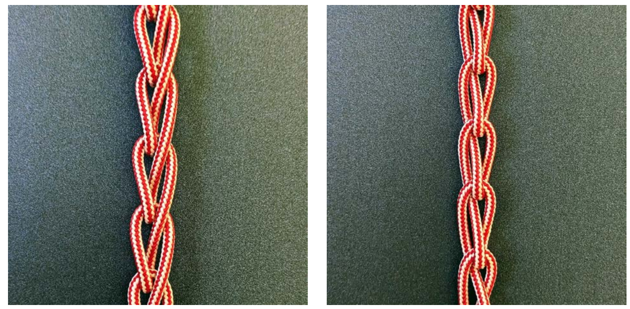
(a) (b) (c) (d)

Figure 1: *Examples of topological crochet.*

Basic Ideas

The starting point of any topological crochet project is to build a ribbon graph from foundation chains. A foundation chain is a braid-like structure that is the first thing a crocheter learns to make. Its front (Figure 2(a)) and back (Figure 2(b)) have different looks. Treating the foundation chains as ribbons, one can set up a ribbon graph by twisting and joining them.

Figure 3 shows the key steps to make the surface in Figure 1(a). We start from the cubic graph with two vertices in Figure 3(a), in the shape of θ. Figure 3(b) is a ribbon graph (the overlapping circles represent the joined stitches at a vertex) whose centerline forms Figure 3(a). Let's call Figure 3(a) the *centerline graph* of Figure 3(b). It results from shrinking the ribbon width of the latter to zero. This is an example of *deformation retract* in algebraic topology [5]. A deformation retract might change the dimension of a space, but it leaves some key features untouched, such as how many independent nontrivial cycles exist in the space. This is an example of *homotopy equivalence*. As explained in [2], the whole surface is a mapping cylinder. These abstract concepts gave birth to topological crochet, but we can successfully crochet a surface without mastering them fully. Crocheting a surface is one of the best ways to understand these concepts.



(a) *Front.* (b) *Back.*

Figure 2: *A foundation chain in crochet.*

(a) *Centerline graph.* (b) *Foundation chain ribbon graph.* (c) *Set up the project.* (d) *Finish the body.*

Figure 3: *Timeline of a trefoil knot Seifert surface.*

Because the graph in Figure 3(a) has an *Eulerian path*, which means a path that visits all edges once, we can build the foundation chain ribbon graph with one long chain, shown in Figure 3(b). Take a foundation chain of 49 (or any preferred number) stitches. Join stitches 49 and 17 with a full 360-degree twist in between them. Then join stitch 1 with stitch 33 so that all three edges connecting the two vertices are twisted 180 degrees in the same direction. The stitches at the same vertex are temporarily held together by stitch markers, which are removed at a later stage. The loop shows where the working yarn end is. Figure 3(b) has right-hand twists, but left-hand ones are done the same way. The result is as in Figure 3(c).

Figure 3(c) is homeomorphic to Figure 1(a), meaning the former is a thinner version of the latter. If the former was made of elastic material, one could stretch it without tearing and gluing to achieve the latter. Instead, we do our "stretching" by adding more rounds along the boundary of Figure 3(c) to make it wider.

Two rounds of carefully designed crochet stitches give rise to the surface in Figure 3(d). We then add a final round over a supportive material (fishing line in this case) to get the final result in Figure 1(a). A workshop was conducted in Bridges 2023 for this project [2]. A detailed step-by-step video tutorial can be found in [4].

This project captures the essence of the philosophy behind topological crochet: we set up a ribbon graph using foundation chains that are homeomorphic to the final surface in mind. The rest of the project will widen the skinny surface where we want.

Having this recipe in mind, we can play with many ribbon graphs. For example, starting with the graph in Figure 3(a), different twist configurations give rise to different surfaces and boundaries. Figure 4 shows another surface based on this graph. By twisting two bands 360 degrees and the third one -180 degrees, we create a punctured Sudanese Möbius band. If we keep crocheting until the boundary closes up, we will have a full Sudanese Möbius band.

Figure 4: *Punctured Sudanese Möbius band.*

Generally, every edge can be twisted any multiple of 180 degrees in either direction. We can represent each twist by an integer. For example, twisting an edge 180 degrees clockwise can be represented by 1 while twisting 360 degrees counterclockwise would be represented by -2. The *parity* pattern for the numbers associated with the twists determines the number of boundary components and the topology of the surface. Table 1 lists all possible surfaces based on the centerline graph in Figure 3(a). One can use Equation (1) to determine the surface types.

Table 1: *Surfaces based on Figure 3*(a)*.*

twists	surfaces
(even, even, even)	3-punctured sphere
(odd, odd, odd)	punctured torus
(even, even, odd)	punctured Möbius band
(odd, odd, even)	punctured Klein bottle

Given the same parity pattern, different twist configurations give rise to different boundary knots or links. For example, Table 2 lists some possibilities for the (even, even, odd) parity pattern. One can determine the boundary knot or link by tracing the edges of the ribbon graphs, or simply making the models.

Table 2: *Boundary of punctured Möbius band.*

# of 180 degrees twists	boundary
$(0, 0, 1)$	two unlinked unknots
$(2, 2, -1)$	two unlinked unknots
$(0, 2, \pm 1)$	Hopf link
$(2, 2, 1)$	Whitehead link
$(2, -2, 1)$	Solomon's knot

Using just the simple graph in Figure 3(a), we see many familiar names from topology appear. One does not need too much convincing to accept that once we dive into the endless sea of all undirected, connected graphs, we can create *every* possible surface, knot, and link. If a centerline graph can be drawn on a surface such that cutting off the graph leaves pieces of discs, we say it is *2-cell embedded* on the surface. For any centerline graph, different twist configurations give rise to all surfaces (with punctures) onto which this graph can be 2-cell embedded. We can close any puncture by shrinking the boundary component to a tiny circle that we can fasten off, similar to crocheting a hat. This way, arbitrary topological surfaces with any number of punctures can be created by finding one such 2-cell embedding. If we start with a centerline graph with V vertices and E edges, the *Euler Characteristic* χ of such a graph is $V - E$. The surface we get is non-orientable if we find a cycle in the ribbon graph along which the overall twist is an odd multiple of 180 degrees. If no such cycle exists, the surface will be orientable. Thus, the parity pattern of the twists is all that is needed to determine if a surface is orientable. The number of boundary components, b, can be determined by tracing along the boundary of the ribbon graph. We can then use the following equation [10] to find the genus g of the surface.

$$\chi = V - E = \begin{cases} 2 - 2g - b, & \text{orientable} \\ 2 - g - b, & \text{nonorientable} \end{cases} \tag{1}$$

For example, every cycle for the ribbon graph in Figure 3(b) has two 180-degree twists, so the surface built from this ribbon graph, shown in Figure 1(a) is orientable. This surface has one boundary, outlined in pale blue yarn. Using Equation (1) we find $g = 1$. Any orientable surface with genus 1 is a torus. Hence, this surface must be a torus with one puncture. This is probably surprising; it takes a lot of imagination to see how a punctured torus can be deformed into this shape. The other three surfaces in Table 1 can be determined similarly. In [10], there is a very detailed discussion of Equation (1) and its application to abstract sculptures.

Simple Designs

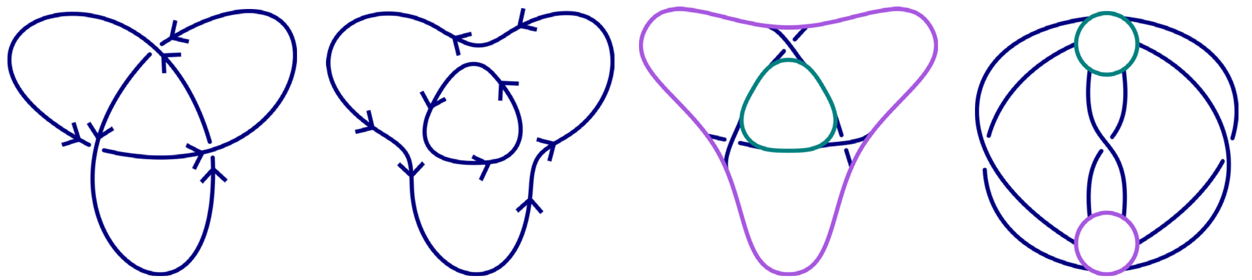
This section focuses on methods for designing a surface with a given knot or link as its boundary.

Seifert's Algorithm

Given a *drawing* of a knot or link where no three strands cross the same point, Seifert's algorithm [12] produces an orientable surface, called a *Seifert surface*, bounded by the knot or link. The steps are as follows:

- Draw an oriented knot or link diagram with arrows.
- Change each knot crossing into parallel strands with the arrows pointing the same way. The diagram now comprises multiple closed loops that don't cross each other.
- Collect all the closed loops; each represents a patch in the surface. The patches are either lying on top of or next to each other. Each patch will be *one vertex* in the foundation chain graph, at which multiple chain stitches join.
- For each crossing in the original diagram, attach a twisted band joining the patches—the final surface results from all patches with all bands joining them. To create a topological crochet model, we make a foundation chain ribbon graph in which each patch is a vertex where chain stitches join, and twisted bands between patches become twisted chains between the corresponding stitches.

Seifert's algorithm takes some practice to master since patches can lie on top of each other in the drawings. Figure 5 shows an example of applying the algorithm to a trefoil knot. In Figure 5(a), we have added arrows in a single direction that trace around the knot. Modifying all three crossings leads to two closed loops in Figure 5(b): outer and inner. These two closed loops are two patches on our surface that bands will connect. We remove the arrows and put all original crossings back in the diagram as in Figure 5(c) to determine how the bands are twisted. These crossings are elongated in Figure 5(d), and the patches are drawn as circles of the same size. The inner patch is moved to the top, and the outer patch to the bottom. The colors match those of Figure 3(b). Each knot crossing becomes a band connecting the two patches with a 180-degree twist. All three bands twist in the same direction due to the symmetry of the knot diagram.



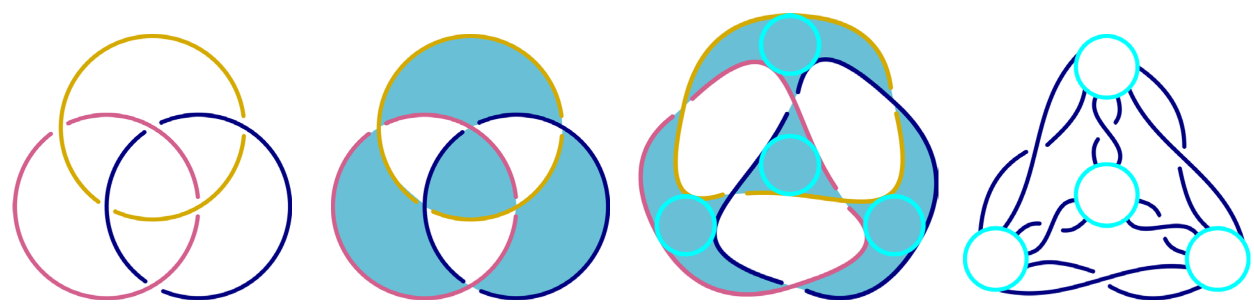
(a) *A knot diagram of a trefoil knot.* (b) *Deformed into two separate loops.* (c) *Crossings are back and arrows removed.* (d) *A side view of the drawing.*

Figure 5: *Application of Seifert's algorithm to a trefoil knot.*

Two-Color Algorithm

Seifert's algorithm only produces orientable surfaces bounded by a given knot or link. Given a drawing of a knot or link in which no three strands cross the same point, the *two-color algorithm* produces two surfaces bounded by the knot or link, which may or may not be orientable. It is also much simpler to apply than Seifert's algorithm. The steps are:

- Draw the knot or link diagram.
- Color the whole paper with a checkerboard of two colors, with the only constraint being that no neighboring regions have the same color. The outer area is also assigned a color.
- Select all the regions with one color. Each region represents a patch on the surface.
- For each crossing in the original diagram, attach a twisted band joining the patches—the final surface results from all patches with all bands joining them. To create a topological crochet model, we make a foundation chain ribbon graph in which each patch is one vertex at which multiple chain stitches join, and twisted bands between patches become twisted chains between the corresponding stitches.

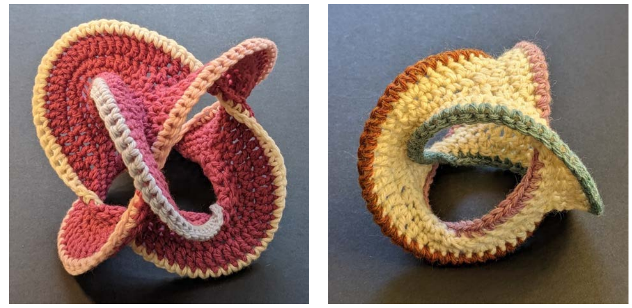


(a) *Link diagram of the Borromean rings.* (b) *Keep one color of the checkerboard coloring.* (c) *Collecting all the patches.* (d) *Foundation chain ribbon graph sketch.*

Figure 6: *Application of the two-color algorithm to the Borromean rings.*

Figure 6 shows how to apply the two-color algorithm to the Borromean rings. Figure 6(b) can be considered as the surface projection on 2-D. Figure 6(c) recognizes four patches that are pairwise connected. The final diagram in Figure 6(d) has four vertices for chain stitches, denoted by four circles in cyan, with six bands, each with a 180-degree twist, joining them. The centerline graph of this ribbon graph is the 1-skeleton of a tetrahedron. The reader can check that had we picked the other coloring, it would also have been a tetrahedron, but with *opposite* twists for each edge. In general, the centerline graphs of the two choices are dual to each other with opposite twists. A detailed step-by-step video tutorial can also be found in [4].

This is a *very* interesting surface. The surface is non-orientable since the ribbon graph has at least one cycle with an odd number of edges (it has many). Since there are three boundary components, $b = 3$. Applying Equation (1), we get $g = 1$, which means it's a thrice-punctured (due to three boundary components) projective plane (because $g = 1$) or twice-punctured Möbius band. Note that the Möbius band is the projective plane with a puncture. *If* we close up all three boundary components with caps, we restore the projective plane. Thus, we have a model that shows *a new immersion of the projective plane in 3-dimensions, which has the chiral tetrahedral symmetry*. This immersion is more symmetric than the most famous immersion of the real projective plane in 3-d, the *Boy's surface*. The Boy's surface can be made from the same centerline graph, with the three edges from one vertex twisted in the opposite direction from the other three edges. Because the parity of the edge twists stays the same, the surface and the number of boundary components stay the same. Adding the three caps will restore the full Boy's surface. Figure 7 shows these two immersions before adding caps. This example is among many that show the power of topological crochet for visualizing abstract surfaces.

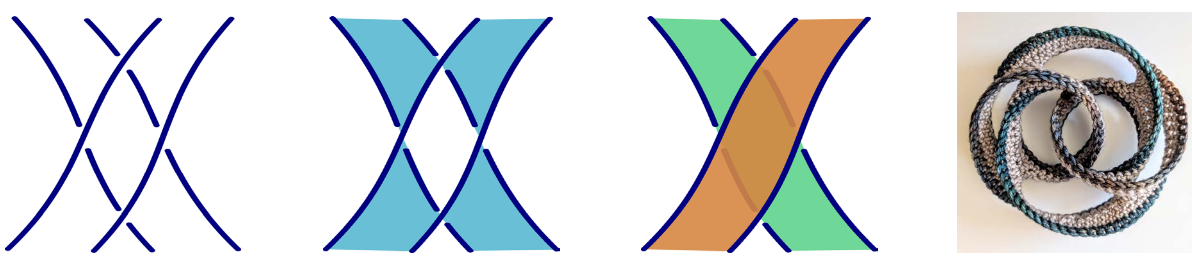


(a) *The new immersion with punctures.* (b) *Punctured Boy's surface*

Figure 7: *Punctured real projective planes associated with tetrahedron.*

Saddle Method And Other Variations

These two algorithms together only give a few among an infinite number of choices of surfaces bounded by a particular knot or link. For example, we can add any number of handles to the surfaces without changing the boundary. Moreover, a single surface can have many variations of its centerline and/or ribbon graph, changing how the surface is made. For example, we can contract some edges to zero length without changing the topology. The *saddle method* is one variation of the two-color algorithm. In [3], we provide a detailed discussion of this method. The surface in Figure 1(d) results from extensive application of the saddle method and was the winner of the *Best Textile, Sculpture, or Other Medium* award in the JMM 2025 Mathematical Art Exhibition.



(a) *Two strands cross over another two strands.* (b) *An original coloring.* (c) *A new coloring with two bands.* (d) *An example designed with this variation.*

Figure 8: *A variation to the two-color algorithm.*

The two-color algorithm only produces ribbon graphs whose centerline graphs are planar with no edge crossings, because all patches don't overlap and no bands cross one another. An important variation of this algorithm allows us to extend it to include surfaces with nonplanar centerline graphs as follows. If a knot or link diagram contains a part that looks like Figure 8(a), and the coloring we choose makes four corners as in Figure 8(b), then replace it with two bands as in Figure 8(c). Two colors are used for the bands in Figure 8(c) to make the diagram clear. Figure 8(d)'s centerline graph has edges crossing/linking each other.

Complex Designs

This section briefly discusses some more complex design ideas that lead to many more interesting surfaces. Details of these ideas will come in future works.

Tangled Graph

Given any ribbon graph, there are infinite ways to arrange it in 3-dimensions. For example, the rhombic triacontahedron has 30 rhombic faces. Take its 1-skeleton and twist all 60 bands while linking one nonadjacent pair for each face, the same way two edges link in Figure 8(d). This extra link gives depth to the project. Without it, a project of this size will have a lot of space inside. With one such link, it becomes Figure 9(a). Different ways of linking lead to different looks.

Emergent Surfaces

We can create a gyroid-like grid to make any *emergent surface*. An emergent surface is where a structure is repeated for a new pattern to appear. For example, Figure 1(b) uses a square grid that has 180 twists everywhere, with the two directions having opposite twists. It is an orientable surface with genus 9 bounded by 12 unlinked unknots. But an annulus appears on a large scale. Figure 9(b) uses a similar grid, and a torus emerges on a large scale.

Self-Intersecting Surfaces

Intersections are everywhere in topology, but are tricky to crochet. But since crocheted surfaces are filled with gaps between the stitches, we are not doomed. The challenge is that the gaps are too small for the yarn balls to go through. One way to solve this problem is to pull the yarn gradually through holes whenever we encounter intersections. But that's extremely tedious, and you can shred the yarn after a few pulls. Alternatively, interlace your fingers from both hands. In the same way that your interlaced fingers alternate at the intersection, we can alternate crocheting rounds for these two surface parts to model the intersection. Figure 1(c) shows a Klein bottle with two punctures made using this method. Figure 9(c) is a variation of this model.

n-Covering

Since our methods are topological instead of geometrical, we can make topological spaces that are not surfaces, for example, the *branched surfaces*. These models look almost like surfaces, except around the centerline graph, where the topology is more complex. This topology can be modeled by crocheting n rounds along each centerline graph edge. For a surface, $n = 2$. Figure 9(d) is an example where $n = 3$. One can see three boundary strands running along each of the edges.

(a) *Linked rhombic triacontahedron.* (b) *Emergent torus.* (c) *Twice-punctured Sudanese Klein bottle.* (d) *5 cell.*

Figure 9: *Complex Designs.*

Conclusion

Topological crochet is an art form that makes algebraic topology concepts real through the work of our fingers. It is fun to design patterns and crochet models that may never have had a physical rendering before.

The ideas of topological crochet can also be applied to other mediums. In principle, it can be applied to any technique that involves coiling, for example, in clay and basketry. Traditionally, the coiling method in clay produces topologically simple objects like pots, plates, and baskets. With skillful hands, a more complex topological surface can also be made using the ideas covered in this paper.

There is no one algorithm to produce a certain surface—rather, infinite possibilities exist, and that's the beauty of this art. Topological crochet can produce models beyond surfaces, such as branched surfaces, and its limit is still being explored. Designing and working on topological crochet projects deepens our understanding of topology and offers alternative forms and mediums within the realm of sculpture..

Acknowledgments

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