The Topology of Movement: Exploring Spatial Orientation through Puppetry

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Abstract

In this workshop, participants build a simple puppet and use it to solve a series of *movement puzzles* in order to explore the topology of real projective space. The movement puzzles involve manipulating the puppet to *perform* paths and *enact* homotopies in projective space in order to gain a visceral understanding of its topology, including the fact that any two non-trivial loops combine to form a trivial loop.

Introduction

This workshop explores *orientation entanglement*, a topological phenomenon that has profound yet subtle consequences for how we move about in space. It is often encountered in one of several popular demonstrations, which include the *plate trick*, the *belt trick,* and the *string trick* [1].

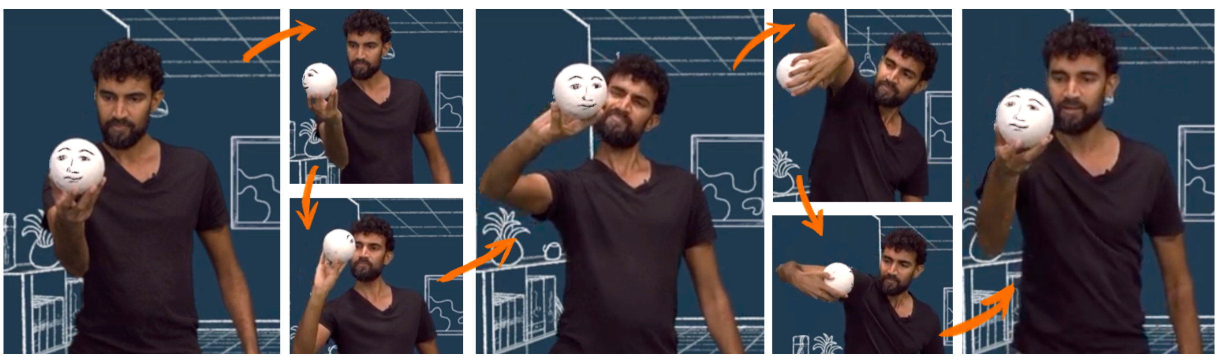


Figure 1: *The plate trick, with Bala the puppet in lieu of the plate.*

All of these *tricks* involve a tethered object that becomes tangled after a single 360° rotation, but returns to its original state after a second 360° rotation. Mathematicians understand this phenomenon as a consequence of the fact that the fundamental group of the real projective space $\mathbb{RP}^3$ is the two element group $\mathbb{Z}/2\mathbb{Z}$. Thus, any loop which represents the group's generator, when *doubled*, becomes trivial and hence homotopic to a point.

Now, the reason this topological fact is relevant to our everyday movements is that $\mathbb{RP}^3$ can be viewed as the set of spatial orientations of a rigid object. Under this interpretation, we refer to $\mathbb{RP}^3$ as *orientation space*.

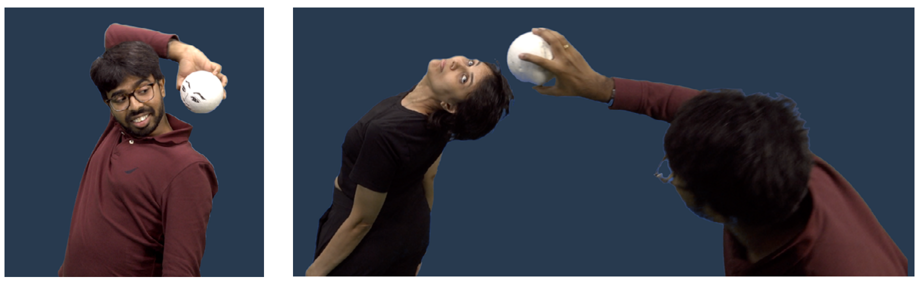
Any 360° rotation of the rigid object can be thought of as a *loop* in orientation space, and represents a generator of its fundamental group. And, perhaps surprisingly, any 720° rotation of the object will be homotopic to a *stationary loop*, which keeps the object fixed in a single spatial orientation.

We have a fairly intimate relationship with orientation space, in that we are highly attuned to the spatial orientation of the various parts of our body, and especially our head. So what would it feel like to experience the homotopy from a 720° rotation to stillness? How exactly does the homotopy work? Can we break it up into smaller pieces? Can we perform it physically with our own body?

This workshop arose as an attempt to answer these questions *intrinsically*, through movement and puppetry, using our inherent spatial intuition to navigate orientation space and understand its topology from within.

This novel approach led to a sequence of movement puzzles employing a simple spherical puppet. These puzzles involve *performing* loops in orientation space as well as *enacting* homotopies of those loops: repeated performances that change slightly each time. Together, these puzzles lead participants to construct and experience the homotopy from a 720° rotation to the stationary loop.

In other words, participants discover why performing a non-trivial loop *twice* is equivalent to remaining stationary. But perhaps the most novel feature is actually the penultimate puzzle, in which we discover that performing two different non-trivial loops *simultaneously* also results in a trivial loop. In fact, our bodies do this all the time in daily life! I first noticed this fact in the process of developing this workshop, and it tends to be a surprising highlight of the workshop, even for participants who have prior experience with these concepts.



(a) (b)

Figure 2: *Two former participants Sharan and Aditi (a) performing a loop, (b) exploring simultaneous loops.*

The workshop is adaptable to a wide variety of audiences, and early versions were conducted for contemporary dancers in Bangalore with limited mathematical background. In fact, one of the key insights for this workshop came from the theater group Kalakshetra Manipur in northeast India, whose performance tradition is based on the idea that figure-eight movements allow us to smoothly transition from movement to stillness.

More generally, I feel the concepts of trivial and non-trivial loops help enrich the movement repertoire for dancers, puppeteers, and anyone interested in creative movement. Dance practitioners may enjoy exploring this alternate way of thinking about movement, which privileges the sense of spatial orientation over the visual sense. At the end of this document I include some prompts that have helped me build movement improvisations around these ideas, and which resulted in the dance piece *Life Cycle of a Trivial Loop* with dancer Joshua Sailo [4].

Finally, I should mention that I have developed an online version of this workshop for the Ministry of Education in India [3], which includes a second part that reframes all the movement puzzles in the language of quaternions, and then creates 3-D animations of them in the software Blender. I have also conducted live versions of this entire sequence as a full-day workshop for college teachers in Tamilnadu.

Workshop Outline

This 90-minute workshop requires oranges (1 per participant) and sharpies (1 per 5 participants). The workshop consists of three parts:

1. Background (20 minutes**):** Participants are introduced to the spherical puppet Bala and the concept of orientation space. We discuss the plate trick and the string trick, and participants build their own versions of Bala.

2. The Movement Puzzles (60 minutes): In order to explore the shape of orientation space, participants are paired off and guided through a sequence of seven movement puzzles of increasing complexity, which involve performing a variety of loops with Bala.
3. Discussion (10 minutes): We return to the string trick and see how it is embodied in the homotopy participants just enacted. We discuss how trivial and non-trivial loops enrich our movement repertoire for dance and creative movement.

Detailed Workshop Instructions

Background (20 minutes)

Our first task is to introduce participants to Bala, the spherical puppet, and to show that Bala's *state* at any moment is determined by both the *position* of his center and his *spatial orientation*. Bala's position is determined by three parameters, corresponding to displacement in the x, y, and z directions. With some guidance, participants figure out that Bala's spatial orientation is also determined by three parameters: two to specify a *direction* for Bala to look, and one to specify an *angle* with which to look in that direction.

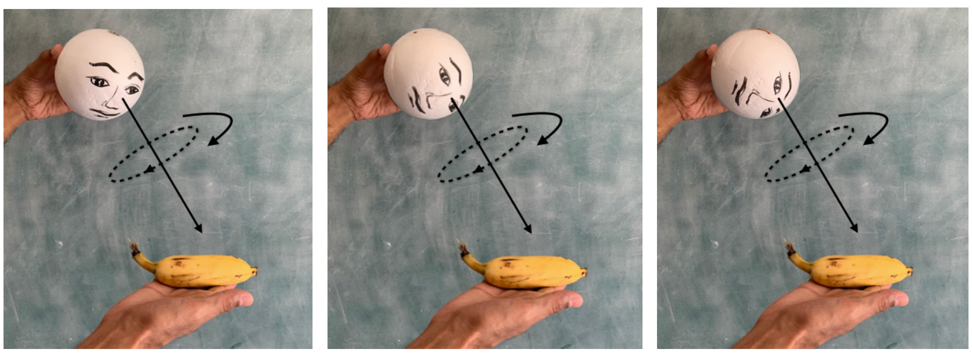
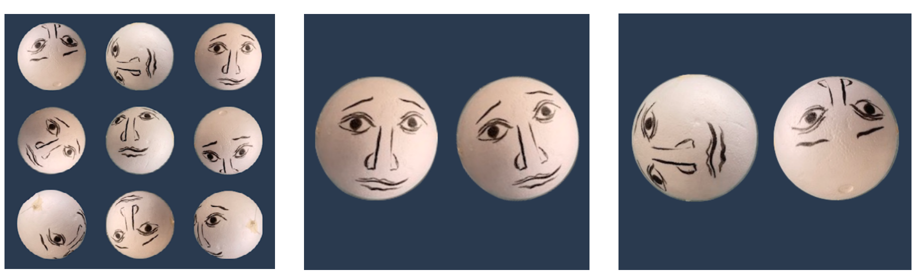


Figure 3: *A full circle of orientations that let Bala look directly at the banana.*

Informally, we can say the set of all spatial orientations is three-dimensional. We now make a crucial observation. The set of orientations is not just some abstract collection, but is in fact a *space*. Each orientation of Bala represents a *point* in this *orientation space*. And we can think of some points as being closer together, and others as being further apart. When we move Bala while keeping his center fixed, we trace out a continuous *path* in orientation space.



(a) (b) (c)

Figure 4: *(a) Examples of points in orientation space, (b) some closer together, (c) some further apart.*

At this point I hand out oranges and sharpies, and have each participant create their own version of Bala, by drawing a face on their orange. This activity may feel silly, but Bala's face will be essential for registering and tracking Bala's spatial orientation.

Although orientation space is three-dimensional, its shape, or *topology*, is very different from that of Euclidean space $\mathbb{R}^3$. For example, we can see that rotations move Bala through orientation space, a bit like how translations move Bala through position space. But rotations behave very differently from translations. For example, if we keep rotating about a fixed axis, Bala returns to his initial orientation. And rotations fail to commute. After fixing an x-y-z coordinate frame for the classroom, participants should check what happens when they perform a 90° x-rotation and a 90° z-rotation, in both possible orders. In [5], Karl Schaffer describes further activities (and dances!) that explore this non-commutativity in more detail.

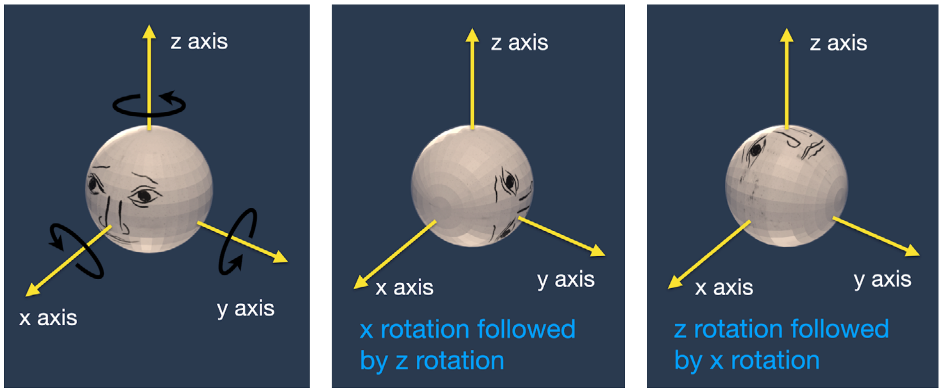


Figure 5: *The directionality of the x, y, and z rotations is indicated by the black arrows. Note that the x-y-z coordinate frame stays fixed as Bala rotates.*

The shape of orientation space governs how our bodies move about in space, with some unexpected consequences. As a demonstration, I show participants the plate trick, and then describe the string trick. We will spend the rest of the session understanding the shape of orientation space, with Bala's help.

The Movement Puzzles (60 minutes)

We now begin a series of seven "movement puzzles" which involve performing *paths* in orientation space, where a path is defined to be a continuous movement of Bala with a well-defined start and end. Participants are paired up, with one person being the *puppeteer*, i.e. the performer, and the other being the *gyroscope*, i.e. the close observer of Bala's spatial orientation. These roles can and should keep switching!

We spend 10 minutes on the first three puzzles. These puzzles are very straightforward but introduce basic terminology and get participants attuned to Bala's spatial orientation.

Puzzle 1 (Perform a Stationary Path): For the very first puzzle, I have participants pick their favorite orientation as a *base point* in orientation space. Bala must perform a *stationary path* at that base point which lasts for exactly five seconds. This ends up being too easy, so I remind them that Bala is performing a path in *orientation space*, so the *position* of his center is irrelevant. Only his orientation must remain stationary. With that in mind, I ask them to perform the most exciting stationary path they can think of (which may involve lightly tossing Bala, passing him back and forth, or even jumping around, provided his spatial orientation stays fixed). The purpose of this puzzle is to emphasize that different movements of Bala will represent the same path in orientation space if their spatial orientations agree at all times.

Puzzle 2 (Perform a Wobble Loop): A *loop* in orientation space is path that begins and ends at the same orientation. The second puzzle is to perform a *wobble loop*. Participants interpret the word 'wobble' however they like, with the understanding that the loop should be somewhat subtle. Typically participants create wobbles that involve Bala's gaze moving in a small circle, a small figure-eight, or a small nod.

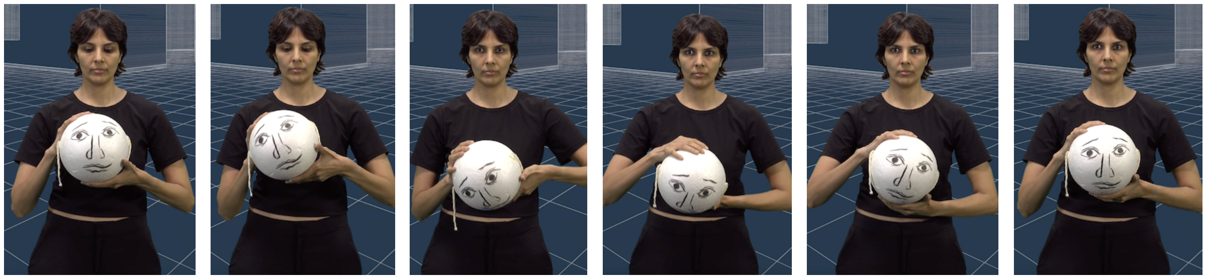
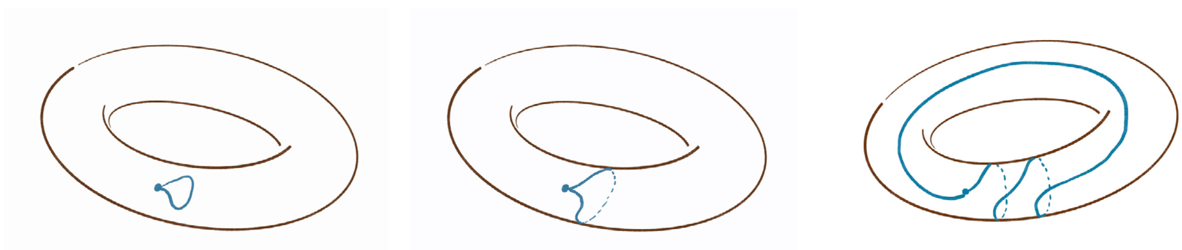


Figure 6: *An example of a wobble loop for the second puzzle.*

*Puzzle 3 (Perform a 360° Rotation Loop)***:** In contrast to the subtle wobble loop, I demonstrate a more extreme 360° rotation loop about the x, y, and z axes. I then ask participants to have Bala perform 360° rotation loops about other more interesting axes. somehow shift the initial axis to the final axis, in a continuous manner.

We spend 20 minutes on the next two puzzles, which involve some actual problem-solving. First, we define a *trivial loop* (in any space) to be a loop that we can gradually deform to a stationary loop. I like to show a couple animations of this on the torus, where it is pretty clear what obstructs non-trivial loops from contracting to a point:



(a) (b) (c)

Figure 7: *Some loops on the torus: (a) trivial, (b) non-trivial, (c) non-trivial. Note that the stationary loop on the torus is a single point, to which only the trivial loop can be smoothly contracted.*

In orientation space, a loop is trivial if we can *enact a homotopy* to the stationary loop: that is, perform the loop repeatedly with minor variations each time, until we reach the stationary loop. Each stage in the homotopy must begin and end at the same fixed base orientation. As an example, I enact a homotopy from the wobble loop to the stationary loop, performing a smaller and smaller version of the wobble until I reach complete stillness.

Puzzle 4 (An Interesting Trivial Loop in Orientation Space): We now fix an axis vector **v**, and consider the loop consisting of a 360° clockwise rotation about **v** followed by a 360° counterclockwise rotation about **v** (using the right-hand rule). The fourth puzzle is to enact a homotopy from this loop to the stationary loop. With some hints, participants usually realize they can keep diminishing the extent of the rotations, until we come to a stationary loop.

Puzzle 5 (Enacting an Equivalence of Rotation Loops): At this point I reveal that any 360° loop is non-trivial: there is no way to gradually deform it to the stationary loop. Although we cannot prove this fact without some algebraic topology, it *feels* believable if we try performing perturbations of the 360° loop. There is some kind of obstruction, which keeps us from contracting it. On the other hand, it turns out that any two 360° rotation loops about *different* axes (say the y-axis and the z-axis) are related by a homotopy. Enacting this homotopy is the goal of this fifth puzzle. If participants are stuck, I note that we need to somehow shift the initial axis to the final axis, in a continuous manner.

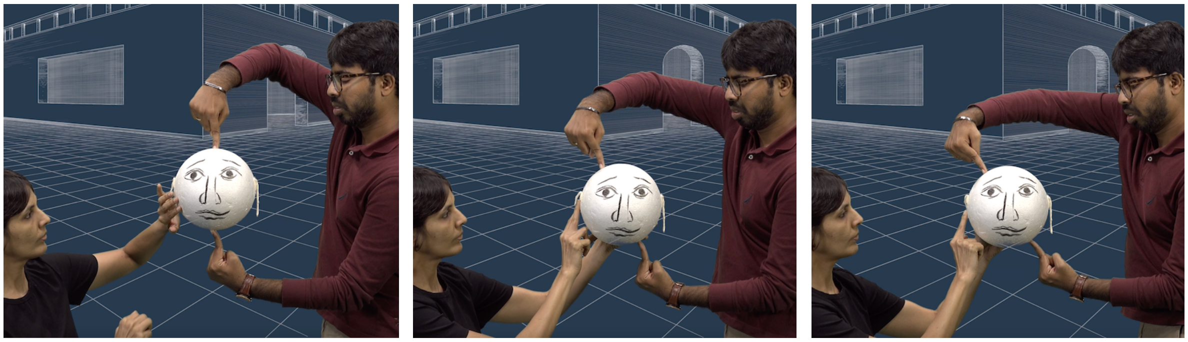


Figure 8: *To solve the fifth puzzle, the axis of rotation must shift slightly in each performance. The base orientation stays constant.*

We spend 30 minutes on the final two puzzles. These puzzles are more involved and investigate the most surprising features of orientation space.

Puzzle 6 (Two Ways to Track an Object): The purpose of this puzzle is to arrive at the realization that *simultaneously* performing two non-trivial loops results in a trivial loop. However, we begin with a seemingly unrelated puzzle: we imagine a star orbiting Bala in the yz-plane, and assign Bala the task of *tracking* the star. In other words, Bala must move in such a way that his blue nose vector is always pointing directly at the orbiting star. (This may involve changing Bala's base orientation, so his initial orientation also points to the object.)

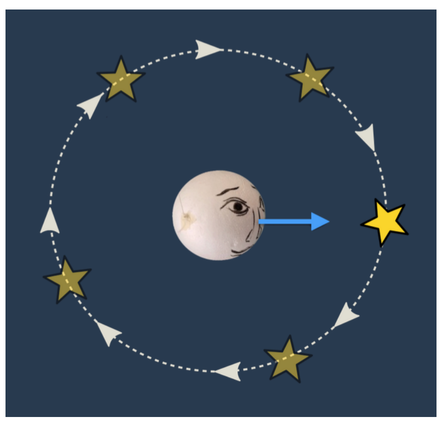


Figure 9: *What loops can Bala perform to track the orbiting star?*

However, there are actually two very different loops that allow Bala to track the star: one trivial and one non-trivial. The sixth puzzle is to find both these loops. The non-trivial loop is simply a 360° rotation loop. The trivial loop is trickier. I like to give the hint: "Use your own head!". Indeed, while the non-trivial loop would be impossible to perform with your own head while keeping your body fixed in place, the trivial loop is the way a human head would track an orbiting object, such as a plane or satellite circling overhead through the sky.

Through follow-up questions, we discover that this trivial loop involves performing two 'impossible' non-trivial loops simultaneously – one where the gaze rotates, and another where the face rotates while keeping the gaze fixed. If anyone is not convinced, they can perform the trivial tracking loop using their smartphone as the orbiting object, letting the phone video-record their face throughout the performance. Playing back the video, they'll see their face indeed performs a full 360° rotation!

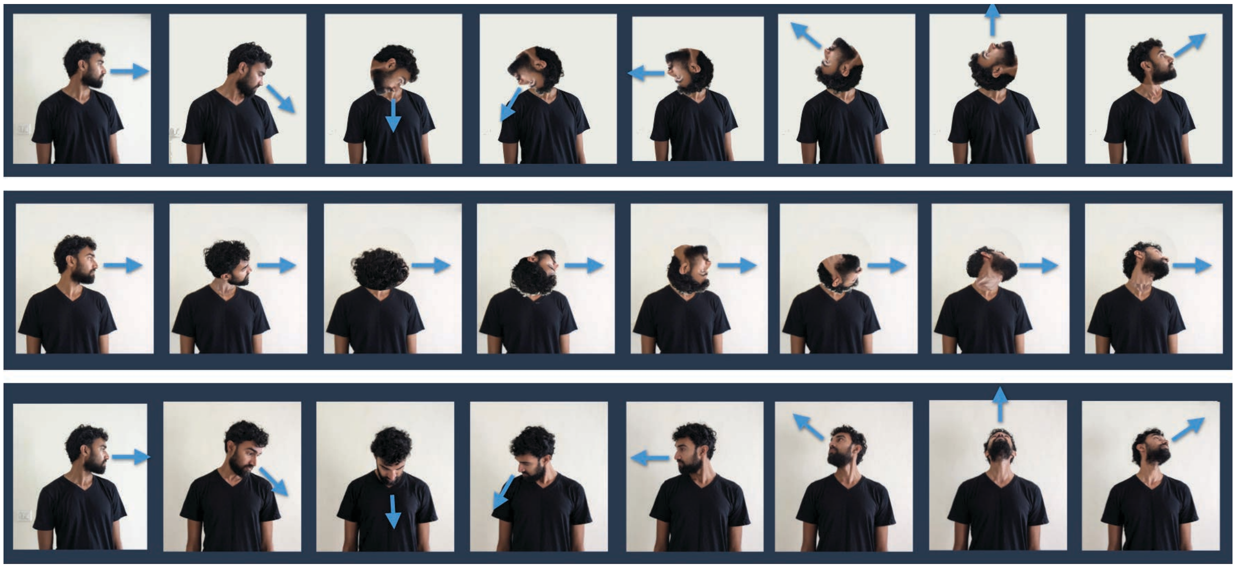


Figure 10: *The top and middle rows represent 'impossible' non-trivial loops. The bottom row is the trivial loop that results from performing these simultaneously.*

Puzzle 7 (The Triviality of the 720° Loop): We've seen that simultaneously performing two non-trivial loops results in a trivial loop. In this final puzzle, we'll see that *concatenating* two non-trivial loops is also trivial.

In particular, I ask participants to make Bala perform a 720° rotation loop with respect to the y-axis, and to find a homotopy of that loop to the trivial loop. The first step is to perform a 720° rotation loop as a single fluid motion. We do this when performing the plate trick, for example, but there the rotation is with respect to the z-axis. In order to perform a 720° y-rotation, it is anatomically easier to let your hand be a *cup* for Bala, and not a *cap*, and to begin with a downward motion. As you finish the first 360° your arm will feel too twisted to continue…but you *can* continue the motion by crossing your arm to the other side of your body. This is essentially the same motion one does while twirling a baton. For helpful videos of this process, see [3].

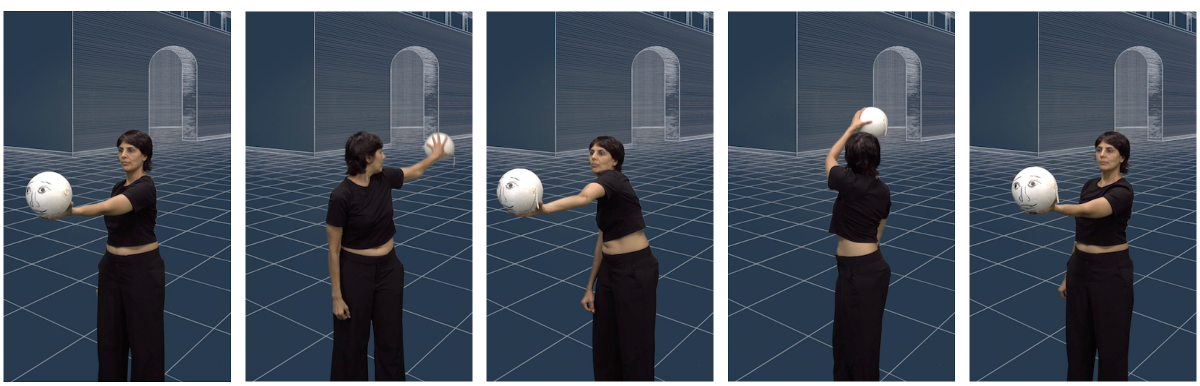


Figure 11: *Performing a single 720° rotation loop with respect to the y-axis. The middle picture depicts the half-way point, when the arm is too twisted to continue, and must cross over.*

Next, participants explore possible homotopies to the stationary loop. Many find that as they perform the 720° y-rotation repeatedly, the axes of the two rotation loops naturally drift apart to create a figure-eight loop, which can then be further drifted (as in the fifth puzzle) into a sequence of two 360° x-rotations in opposite directions. Finally, using the solution to the fourth puzzle, we arrive at the stationary loop.

Discussion

To wrap up, we return to the string trick, which is beautifully captured in a 3-D animation by Jason Hise [2]. Watching the animation together, we observe the homotopy participants enacted in the seventh puzzle. In particular, the innermost sphere in the animation is stationary, while the outermost sphere performs a 720° z-rotation loop. Meanwhile the intermediate spheres perform various stages in the homotopy.

I like to end with a discussion of how the framework of trivial and non-trivial loops can enrich participants' movement repertoire. The homotopy they discovered in the seventh movement puzzle allows us to smoothly transition from movement to stillness, and from stillness back to movement. To explore this idea further, I've found the following prompts helpful:

- *Can you find examples of smooth transitions between movement and stillness in the world around us? Do figure-eight motions play a role? (Hint: watch a bird flap its wings in slow motion!)*
- *Can you perform 720° x-rotations and z-rotations, and smoothy transition these to stillness?*
- *Can you find more examples of motions that involve* simultaneously *performing non-trivial loops about different axes?*
- *And finally, can we devise a short dance exploration around these ideas, prioritizing your sense of spatial orientation over your visual sense?*

Conclusion

Orientation entanglement is a rare example of a topic that can be approached both through movement exploration and through mathematical formalism. The aim of this workshop is to bridge these approaches, creating a space where movement exploration deepens mathematical understanding, while a mathematical framework enlivens our movement repertoire.

The workshop can be extended to allow for further exploration, either on the mathematical side, or the artistic side. For the former, I've created an online follow-up to the movement puzzles which explores the quaternion number system for coding 3-D animation in Blender [3]. For the latter, contemporary dancer Joshua Sailo and I have an ongoing collaboration, *Life Cycle of a Trivial Loop* [4].

I believe our sense of spatial orientation can be a potent source for creative expression in movement-based art, including puppetry and dance. I hope this workshop allows for mathematicians and artists to connect with this rather primal physical sense.

Acknowledgements

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