Exploring an Open Question About Magic Squares and Classical Labyrinth Seed Patterns

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Abstract

This workshop explores labyrinths, labyrinth seeds and magic squares and their possible structural relationships. We share new and established ideas about the mathematical/ artistic patterns underlying classical labyrinths and magic squares, and engage in hands-on explorations of these fascinating forms, through drawing, coloring, construction, movement, dance and discussion. Each of the coauthors of this workshop paper has worked with classical labyrinths and magic squares with our own students and with whole intergenerational communities and artists; we bring this mathematical and artistic experience to the design of this workshop.

Prelude

This inquiry began with our reading of a startling observation by scholar Robert Ferré [8]:

> There is another aspect of the seed pattern that I find mind boggling. It has to do with magic squares... In a magic square, each column and row add up to the exact same sum, as do the diagonals... Many of them were assigned the names of heavenly bodies. *The one I want to describe is the Square of the Moon [degree 9]... Suppose we mark all the odd numbers in the magic square. What happens? We get the seed pattern for a classical labyrinth! I have no idea why or how that happens. All magic squares with an odd number of squares, 5x5, 7x7, 9x9, etc., exhibit this phenomenon* [or so Ferré claims: SG] *Those with an even number of squares such as 6x6, 8x8, do not form a seed pattern. Instead, they make a checkerboard pattern, which I find no less puzzling.* (pp. 43–44) [Italics added.]

Ferré's observations and conjectures/claims are also published in the English edition of Kern's authoritative volume [10], p. 38–39. But are these claims accurate? And if they are, why would this be so? Do magic squares have a necessary structural relationship to classical labyrinths and their geometric seed patterns? If such a relationship exists, which came first: the labyrinths, the seeds or the magic squares?

This paper and workshop address these questions through participants' experiential engagement in activities designed to help people understand and compare the structure of classical labyrinths, their geometric seeds and magic squares. We will introduce and (collaboratively) create large-scale, walkable classical labyrinths, walking and 'dancing' them to understand their structure. We will play with the numerical patterns of magic squares and investigate open questions about their relationship to labyrinth seeds, aiming to create new conjectures about these fascinating patterned objects and processes.

Introduction to Labyrinths and Labyrinth Seeds

Labyrinths are an ancient form with a history that goes back thousands of years, in places as far apart as Indonesia and the southwestern US, Siberia and Sweden, north Africa and the Middle East [12]. Most of what we know of the history of labyrinths comes to us via oral history and archaeological material records, so that the history of their design and use is partly conjectural. Nonetheless, the distinctive form of the 7-course unicursal classical labyrinth, with a single pathway leading to and from the central goal, continues to be impressive and fascinating. Although other designs exist, we are limiting our scope here to the classical labyrinth, the oldest and most widespread design, because of its relationship to geometric labyrinth seeds. A number of previous Bridges papers have dealt with the mathematical and artistic aspects of labyrinths [1,5,7,15], and we have drawn on some of these findings in this workshop paper, including considerations of labyrinth seeds and their variations [1,15], and alternative methods of generating labyrinths [1,5,7].

Figure 1: *Restored 15th century fishermen's labyrinth and interpretive signage, Landsort (Oja), Sweden. Photos: Gerofsky*

The most common labyrinth seed is a line drawing: horizontal and vertical lines radiating from a point, with four angled 'elbow' shapes rotated in the four quadrants, and a dot in each of the four corners. When connected with arcs in pairs starting with two contiguous points or ends of lines, this seed produces a 7-course classical unicursal (one-path) labyrinth, with a single entrance/ exit opening, and a path that winds in non-obvious, sometimes surprising ways towards a central goal area (see Figs. 2a and b).

If we number the seven courses of this classical labyrinth from the outer (#1) to the inner one (#7), the route we take by following the unicursal pathway follows the courses in this order: start⟶ 3, 2, 1, 4, 7, 6, 5 ⟶ goal. This makes the classical labyrinth less intuitively predictable than a simple spiral path that leads directly from the outer loop through each inner loop to the goal. Walking the classical labyrinth, one has the impression of moving away from the goal in the first three rounds (3, 2, 1), then coming tantalizingly close to the goal (4, 7) but being not able to enter it, and finally, somewhat unexpectedly, entering the goal from course 5 (6, 5). Some labyrinth enthusiasts view the labyrinth walk as an analogy for a person's path in life, and certainly these surprise twists and turns, seemingly moving away from the goal and suddenly coming upon it, is a phenomenon many people experience in life.

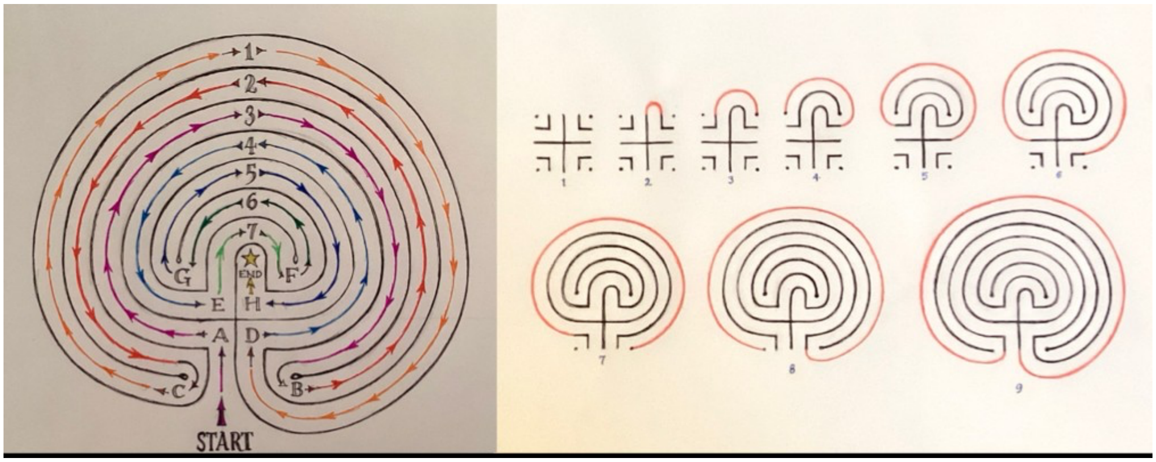


Figure 2: (a) *7-course classical labyrinth with route marked;* (b) *constructing a 7-course labyrinth from its seed. Illustration: Lambourne*

In this workshop, we explore variations on this common labyrinth seed by playing with variance and invariance: holding most features of the seed invariant, and then systematically varying one feature, to see if the resulting new seed still produces a walkable labyrinth, and to notice how its structure varies from the classical 7-course unicursal labyrinth. Features that might be varied include symmetries of the seed, number of dots, number of angled L-shapes (which we name as *elbows*), ways of connecting the dots and line termini, and so on (see Fig. 3).

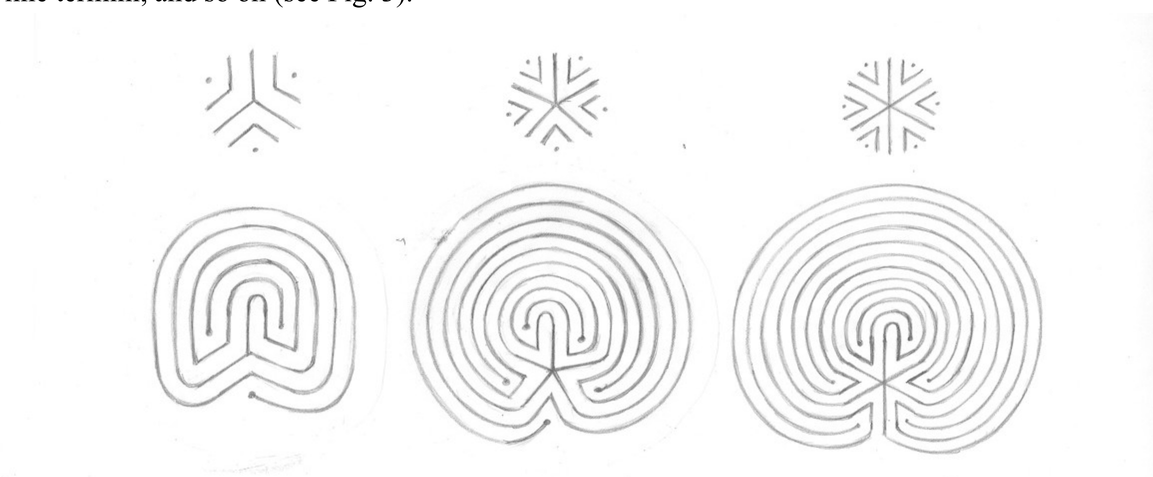


Figure 3: *Experimenting with variations on labyrinth seed symmetries. Illustration: Lambourne*

Introduction to Magic Squares

Magic squares, like labyrinths, are mathematical/ artistic phenomena that have been explored in many cultures from ancient times. A magic square of order n is an $n \ x \ n$ square grid filled with the distinct whole numbers from 1 to n^2, in which every row, column and main diagonal adds to a constant sum. Some magic squares, like the one on the facade of Gaudi's Sagrada Familia cathedral in Barcelona (where the sum is 33, related to Jesus' age at death) are considered less mathematically interesting because they include repeated digits in a row or column. (In contrast, Latin squares, related historically to magic squares, do have n repeated digits in an $n \ x \ n$ square, and do not have the constraint of diagonals adding to the 'magic' sum.)

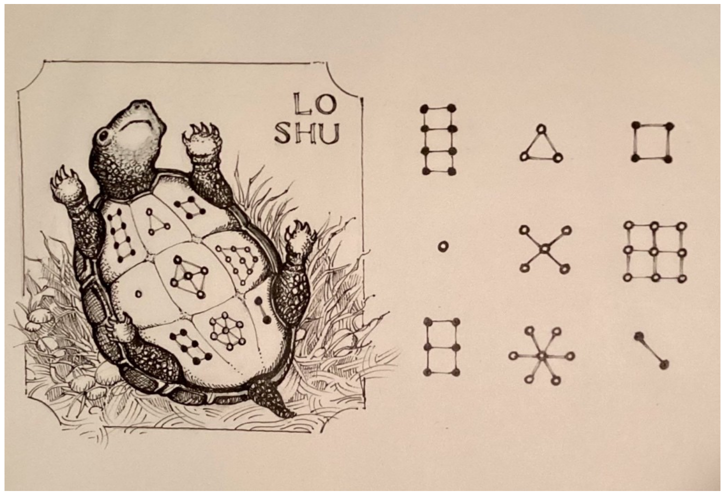


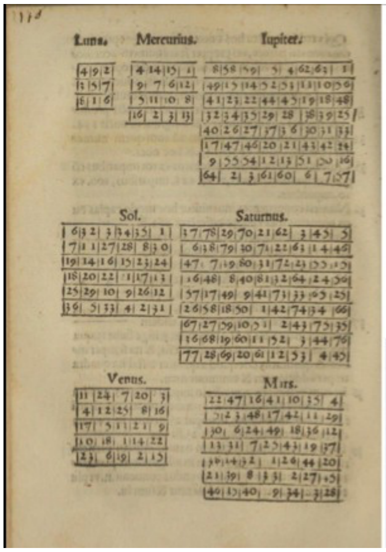
Figure 4: *The Lo Shu magic square. Illustration: Lambourne*

The earliest magic squares on record are from Chinese written records dating to 300 BCE [13,14] and dating back to earlier times. According to Chinese legend, the first magic square, of order 3, was discovered on the plastron (undershell) of a turtle emerging from a flood of the Lo River, c. 3000 BCE (See Fig. 4). The Lo Shu magic square was reputed to have been used in divinatory practices to control the river's flooding. Records of magic squares of order 3 and larger have been sourced from India (587 CE), Baghdad (983 CE), and throughout East Asia, the Islamic world, and eventually Europe, though Sesiano suggests ancient Greek sources as well [13]. From the earliest times, the 'magic' in magic squares referred both to their mathematical beauty and to their use in divination.

Much of the mathematical research on magic squares has dealt with methods of construction and of trying to enumerate the number of possible magic squares of each order n. For example, John Conway's Bridges 2019 plenary talk (similar to [4]) explored algorithms for producing magic squares. Surprisingly, the number of possible magic squares of order $n > 6$ is still an open, unsolved mathematical question, while the exact number for $n = 6$ was discovered as recently February 2024 by H. Mino, Professor Emeritus at the University of Yamanashi, Japan [10]. The number of magic squares (up to rotation and reflection) for lower orders of n, beginning with $n = 3$, are: 1 $(n = 3)$, 880 $(n = 4)$, 275,305,224 $(n = 5)$, and 1,775,388,9197,660,635,632 $(n = 6)$ [6, 16]. A small number of previous Bridges papers have also dealt with aspects of magic squares, and we have drawn from one of these [17] on the early history of magic squares.

The order 3 magic square was associated with child-bearing in both India circa 900 CE and the Middle East circa 700 CE. Magic squares of order $n > 3$ were studied extensively throughout medieval Islam. Interestingly, there is a tradition of associating seven particular magic squares of orders 3 to 9 with celestial bodies — the same celestial bodies that we use to name the days of the week. (Conway references these in his G4G talk as well [4]). These magic squares are associated with the following 'planets' or celestial bodies: $n = 3$: Saturn, $n = 4$: Jupiter, $n = 5$: Mars, $n = 6$: the Sun, $n = 7$: Venus, $n = 8$: Mercury, and $n = 9$: the Moon. Comparing these numbers with the naming of the days of the week, starting from Sunday, we have the sequence 6 (Sun), 9 (Moon), 5 (Mars), 8 (Mercury), 4 (Jupiter), 7 (Venus), and 3 (Saturn). Note that 15th- 16th century Italian mathematicians Pacioli and Cardano [2] both published these planetary magic squares, but apparently matched the squares to celestial bodies in a non-standard way (see Fig. 5).

According to Comes [3], the relationship to planetary bodies first appeared in ibn Zarkali's (1029-1087) work Kitāb tadbīrāt al-kawākib (Book on the Influences of the Planets), and by the end of the 15th century the planetary magic squares had become widely known in Europe. Sesiano's [13] 2019 text provides a thorough history of the rise of magic squares in both mathematics and early occult traditions.



Saturn=15

4	9	2
3	5	7
8	1	6

Jupiter=34

4	14	15	1
9	7	6	12
5	11	10	8
16	2	3	13

Mars=65

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

Sol=111

6	32	3	34	35	1
7	11	27	28	8	30
19	14	16	15	23	24
18	20	22	21	17	13
25	29	10	9	26	12
36	5	33	4	2	31

Venus=175

22	47	16	41	10	35	4
5	23	48	17	42	11	29
30	6	24	49	18	36	12
13	31	7	25	43	19	37
38	14	32	1	26	44	20
21	39	8	33	2	27	45
46	15	40	9	34	3	28

Mercury=260

8	58	59	5	4	62	63	1
49	15	14	52	53	11	10	56
41	23	22	44	45	19	18	48
32	34	35	29	28	38	39	25
40	26	27	37	36	30	31	33
17	47	46	20	21	43	42	24
9	55	54	12	13	51	50	16
64	2	3	61	60	6	7	57

Luna=369

37	78	29	70	21	62	13	54	5
6	38	79	30	71	22	63	14	46
47	7	39	80	31	72	23	55	15
16	48	8	40	81	32	64	24	56
57	17	49	9	41	73	33	65	25
26	58	18	50	1	42	74	34	66
67	27	59	10	51	2	43	75	35
36	68	19	60	11	52	3	44	76
77	28	69	20	61	12	53	4	45

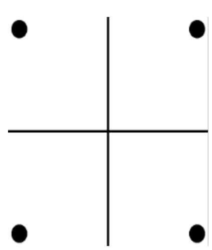
Figure 5: *The seven 'planetary' magic squares from Cardano[2] and a more standard attribution.*

Do (All) Odd-order Magic Squares Generate Labyrinth Seeds?

We can state two conclusions about this question with assurance: The planetary magic squares of order $n = 5$ and $n = 9$ generate classical labyrinth seeds (see below). But it is important to note that there exists a large number of distinct odd-order magic squares (for example, 275,305,224 squares of order 5), and we can offer plenty of counterexamples from this inventory that show that *not all* odd-order magic squares of $n = 4k + 1$ generate labyrinth seeds.

What remains unanswered is whether there is a whole class of odd-order magic squares that generate classical labyrinth seeds (through a particular coloring), and why it should be that the construction of the planetary magic squares in particular generates labyrinth seeds. The first remains an open question because there have so far been no criteria identified for a comprehensive classification of magic squares of order n.

Let's turn our attention to the classical labyrinth seed which includes four radial lines at right angles, and four dots, and may also include $4k, k > 0$ (symmetric) elbows. It is possible to find these labyrinth seeds within some magic squares by simply coloring all cells with odd numbers, as Ferré suggests [8].



23	6	19	2	15
10	18	1	14	22
17	5	13	21	9
4	12	25	8	16
11	24	7	20	3

Figure 6: *The labyrinth seed emerges from the 5x5 planetary magic square by coloring odd numbers.*

Before we delve into the constructions of such magic squares, we can make a few observations: First, classical labyrinth seeds emerge only from magic squares of order $n = 4k + 1$, for $k = 1,2,3,\ldots$ To demonstrate this, we will define the elements of the classical labyrinth seeds which are joined systematically to create the labyrinth as follows (see Fig. 2b above to follow the drawing process):

A classical labyrinth seed is composed of m radial lines (most typically, $m = 4,$); km optional 'elbows' (i.e., V-shapes that are placed between each pair of radial lines, and where $k > 1$, the elbows can be stacked, with a space between); and m non-optional dots that are placed at the vertices of an imagined symmetrical m-gon that could be drawn through the line termini of the seed. The m radial lines and m dots are necessary components for a classical labyrinth seed, since the radial lines form the internal boundaries that give the completed labyrinth its entrance point, goal and bounding lines, and the dots create 180º turning points that are a necessary part of the labyrinth form. If the optional elbows are omitted, the labyrinth will have fewer turns, but will still function as a labyrinth. However, if the dots are omitted, the resulting shape will not be a labyrinth, as it will consist of number of closed paths that cannot be entered or exited.

A *line terminus* (L) is the point at an end of one of the radial line segments or elbows.

A *dot* (D) is one of the points or dots located symmetrically at the corners of the labyrinth seed. (i.e,, the corners of the imagined m-gon that could be drawn around the seed).

The classical labyrinth is drawn by connecting a pair of adjacent line termini, or line terminus and dot, and then subsequently connecting all the other pairs moving outward and connecting each subsequent pair, as shown in Fig. 2b.

A *path* (P) is the 'walkable' gap between two lines created when pairs of line termini, or a line terminus and a dot, are connected.

The *exterior border* of the labyrinth seed is the imagined edge of the symmetrical m-gon that could be drawn through the line termini of the seed. By giving attention to the exterior border, we can count the number of line termini (L), dots (D) and paths (P) that are necessary to create the classical labyrinth from its seed.

It should be clear that n must be odd because the exterior border of a labyrinth seed has an odd number of line termini and an even number of paths. Furthermore, the addition of each additional m elbows to the labyrinth seed results in two additional line termini (L) and two terminal paths (P) on each edge of the seed's exterior border. Each border begins and ends with a dot (D), and so the first few magic squares that can house a labyrinth seed have borders DP**L**PD, DPLP**L**PLPD, and DPLPLP**L**PLPLPD, and are of order 5, 9, and 13, respectively.

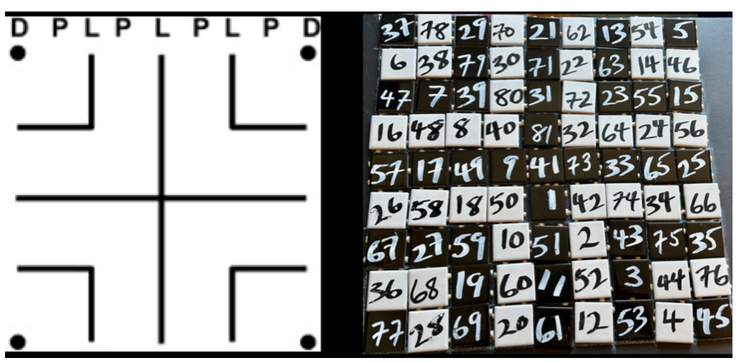


Figure 7: *An example (order 9) showing why a magic square housing the seed of a classical labyrinth must be doubly even plus one.*

The four planetary magic squares of odd order which might potentially house the seed of a classical labyrinth can be generated by a variation of the so-called Siamese method, which normally begins by placing a 1 in the middle of the top row and then filling in successive numbers diagonally southeast. Once a number is placed outside the square, we imagine the square as a torus and place the number in the associated position. In the Siamese method, if a cell is blocked, the number is simply placed in the next available cell below. The *planetary* type of odd magic square has the 1 in the cell just below the center. We will generate a planetary type of magic square using a variation of the Siamese method: when we encounter a block we shift two squares down.

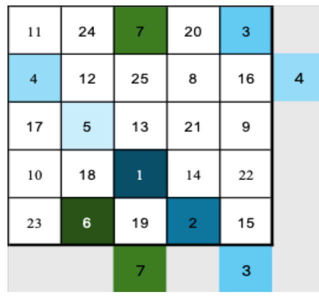


Figure 8: *Generating a planetary magic square with a variation of the Siamese method.*

There are other magic squares of order $n = 4k + 1$ that generate classical labyrinth seeds, though there does not yet appear to be any classification of such a group. Although there are many transformations of magic squares that also produce a magic square, for example swapping two rows, our investigation thus far shows that most magic squares of odd order cannot be transformed into one which holds a labyrinth seed. We have found that, for a magic square of order 9, the method of superposition, followed by a series of transformations, also produces a labyrinth seed. It is likely that this method will also work for all $n = 4k + 1, k > 2$. Our first record of the method of superposition comes from the book *Ganita Kaumudi* (1356), written by the Indian mathematician Narayana Pandit, and it is a method which has been studied and built upon by many others, including Euler.

Venus, the planetary magic square of order seven, does contain a shape similar to those of the other odd planetary magic squares by coloring the odd numbers, though without the dots. As noted above, this shape, made of lines and elbows but no dots, does not generate a labyrinth. Similarly, the Saturn planetary magic square of order 3 contains the radial lines, but no dots (or elbows), and thus is not a labyrinth seed.

Workshop Timelines, Materials and Description

Our hands-on, experiential workshop introduces participants to classical labyrinths and their seeds as well as to magic squares, and offering a chance to experiment with magic square colorings to explore the open questions about potential relationships between magic squares and labyrinth seeds:

1. We will start with an introduction to collaborative drawing and varying classical labyrinths starting from geometric seeds and playing with variance/ invariance. For example, we will ask participants to experiment with labyrinth seeds with other than four-fold symmetries (some of which are illustrated in Fig. 3 above), and with varying the number of elbows and dots.

2. We will introduce constructing magic squares by the Siamese, superposition and other techniques, using playing cards, painted tiles and paper and pencils, and then color printed planetary and other magic squares to highlight emergent patterns created by the odd numbers on the planetary magic squares.

3. We will make and walk variations on classical labyrinths using sidewalk chalk or masking tape (outdoors on pavement, or indoors on carpet or tarp.) After walking the labyrinth individually, we will teach participants how to do Gardner's Double Appleton labyrinth dance (as shown with dancers and an animated diagram in [9]) to explore further structural features of classical labyrinths. This labyrinth dance allows for two or three people to walk the classical labyrinth arm in arm, even though some will be in the process of entering the labyrinth and others in the process of exiting. The collaborative action of 'dancing' the labyrinth in this coordinated way highlights elements of the structure of the labyrinth more difficult to notice in an individual walk; for example, dancers differentiate the 'hotspots' where a new person enters or exits the dance, or where a person connects with others exiting the labyrinth goal. This activity gives

salience to an embodied, movement-oriented collaborative approach to researching geometric forms.

4. Finally, we will explore open questions about labyrinth seeds and magic squares together. There is a structural correspondence between the seeds and the $n = 4k + 1$ planetary magic squares of order n, but we have not yet encountered nor derived a proof for why this might be a necessary result (or, conversely, a proof that it might simply be coincidence!) We will enlist the collective efforts of the workshop participants to help advance work on this open question in future.

Summary and Conclusions

We introduce aspects of the history and mathematics of classical labyrinths, geometric labyrinth seeds and magic squares and their possible connections in this paper. Participants in this Bridges workshop gain acquaintance with the beauty and mathematical structure of these fascinating historical forms through drawing, coloring, constructing, moving individually and collaboratively, and discussing. A provocative claim about a potential relationship between magic squares and labyrinth seeds is explored, and we are able to take further steps in reframing some fascinating open questions about this relationship.

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