Icosahedral Quasicrystal Folded Paper Strip Spheres

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Abstract

A new method for building icosahedral quasicrystal structures out of paper is explored and described. Three-dimensional quasicrystal patterns arranged in spherical formations can be modeled using sets of folded paper strips that are joined into loops. Each strip consists of a sequence of golden rhombi, separated by parallel mountain and valley folds that may be pre-scored. The loops overlap and are connected by tape such that corresponding folds in different strips reinforce one another to form a rigid structure. These sphere-shaped polyhedra are nonconvex but share similarities with zonohedra. We demonstrate how the strips appear both in their two-dimensional unrolled state and in their completed construction. Interesting patterns can be observed by comparing icosahedrally symmetric designs with those that are less regular. These differences are related to the underlying choice of tiling units that are used to generate the packing and resulting surface design.

Introduction

The icosahedral packings of quasicrystals can be seen as the 3D equivalent of the 2D aperiodic Penrose tiling. We may construct 3D packings with two rhombohedral tiles (Figure 1, left) that fill space without any gaps. These tiles are the *acute rhombohedron* A_6 and the *oblate rhombohedron* O_6. Steinhardt et al. [3] [5] proposed a theoretical description for quasicrystal construction that consists of 4 distinct tiles (Figure 1, right) rather than the two rhombohedra. All are golden isozonohedra. The acute rhombohedron A_6 is one of the four tiles. The other three tiles are the *rhombic (Bilinksi) dodecahedron* B_{12}, the *rhombic icosahedron* F_{20}, and the *rhombic triacontahedron* K_{30}. Note that the three larger zonohedra may be also subdivided into packings of only A_6 and O_6 tiles.

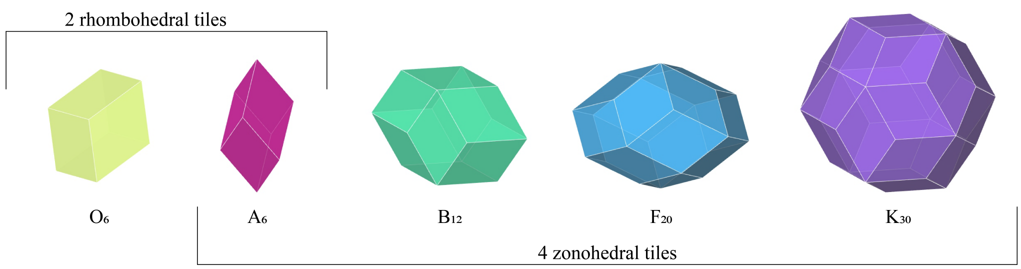


Figure 1: *The five golden isozonohedra. The first two, O_6 and A_6, are the two types of rhombohedra that form rhombohedral space-filling quasicrystal packings. The last four, including A_6, are the four zonohedra that may also form space-filling quasicrystal packings.*

Although quasicrystals have been studied extensively for decades, there does not yet exist an accessible way to generate and visualize these structures. In 2023, I submitted a video [1] to the Bridges Short Film Festival demonstrating 3D quasicrystal construction using a substitution algorithm by Alexey E. Madison [4]. Around the same time, I was also implementing that algorithm as a flexible software tool for Grasshopper 3D that could easily construct these packings to fill any given boundary volume. This tool both enabled the generation of the animations in the film, and also provided a way to further explore and examine these structures in a digital environment, thus allowing for the present study. For this paper, we focus specifically on spherical configurations, treating the outer boundary of a quasicrystal packing as a single closed polyhedron.

Quasicrystal Spheres

We begin by defining a set of nonconvex polyhedra generated from quasicrystal packings that we call *quasicrystal spheres*. Using the four zonohedral tiles, there exist exactly three complete packings with a single center of icosahedral point symmetry that completely fill 3D Euclidean space (Steinhardt [5] and Madison [4]). Here we choose to focus on one of these options, which features a single K_{30} tile at the center of the pattern. Using this same center point, we generate a sphere of any size and trim the infinite tiling by this sphere, taking only the tiles whose centerpoints fall within the spherical boundary. Next, we abandon the configuration of the inner tiles to focus only on the outer surface formed by the exposed faces of the outermost tiles. Together, these faces join to form a single non-convex polyhedron, or quasicrystal sphere. Examples of these are shown in Figure 2.

Despite their nonconvexity, these polyhedra have several regular properties. Their faces are all congruent golden rhombi used in all faces of the original 3D unit tiles. They exhibit icosahedral symmetry, ensured by the icosahedral symmetry of the underlying tiling (and the spherical boundary, which maintains that symmetry). All edges of the quasicrystal spheres are positioned in one of six possible orientation directions, corresponding with the six 5-fold rotational axes of the icosahedron. This property is shared by the rhombic triacontahedron (the K_{30} tile). Quasicrystal spheres share similar properties with zonohedra, in that their faces can be grouped into encircling bands of faces sharing a common edge direction.

By starting at an arbitrary face and picking one of its edges, then moving successively through neighboring faces sharing that edge direction, we can visually observe that we will always follow a continuous band that returns to the starting face. Every face corresponds to a crossing between two distinct bands. Unlike zonohedra, however, there are more than one band of faces for any given edge direction.

Paper Strip Analysis and Construction

While both 3D printing and Zometool [6] sets exist as available options for constructing physical models of quasicrystals, the zonohedra-like properties of quasicrystal spheres offer a new possible construction method using continuous strips of paper or other flat material.

George Hart's study on zonohedrification [2] inspired a simple algorithmic method for extracting the groups of bands on any given quasicrystal sphere corresponding to a single edge direction (what would be a single band, or zone, for a true zonohedron). We start by choosing one of the six known edge direction vectors. Next, for each face, we compare the normal vector with this edge direction vector. If the dot product is zero, we know that this face belongs to a strip (or group of strips) where all interior edges are in the chosen direction.

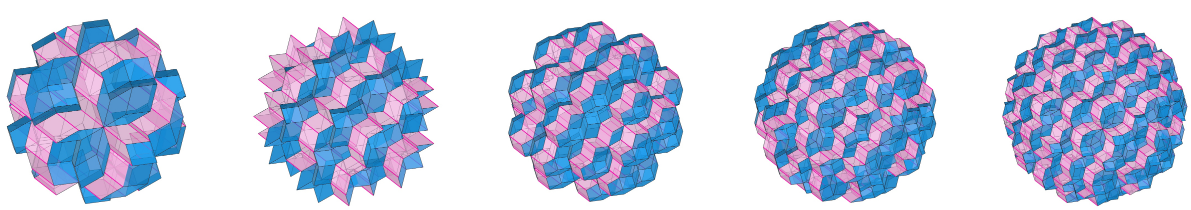


Figure 2: *A selection of quasicrystal spheres rendered in partial transparency. On each sphere, one zone group, consisting of parallel edges and bands, is highlighted in pink. All other faces are blue.*

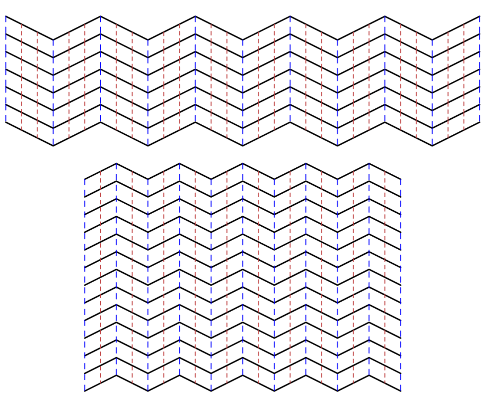
We apply this method for arbitrarily large quasicrystal spheres and observe how the size and number of strips increase, while the continuity of the strip loops are maintained (Figure 2). The number of these bands is always odd. Thus, every zone group contains a primary central band that wraps around the middle of the polyhedra, with pairs of smaller bands wrapping parallel to the central band on either side.

Given that the quasicrystal spheres, as we have considered them so far, exhibit icosahedral symmetry, each of the six groups of bands are identical. This makes it easy to construct the 2D cut-and-score sheets for the construction of the paper model. First, we determine a single edge direction group and unroll the bands of surfaces, determining the necessary mountain and valley folds in the process. We then take exactly six copies of the defined bands. We may also simplify the model by only choosing specific strips per each group, rather than all strips, leaving a pattern of holes in the built model (Figure 4) where two missing strips would have overlapped. This also leaves some single surfaces not belonging to an overlapping pair.

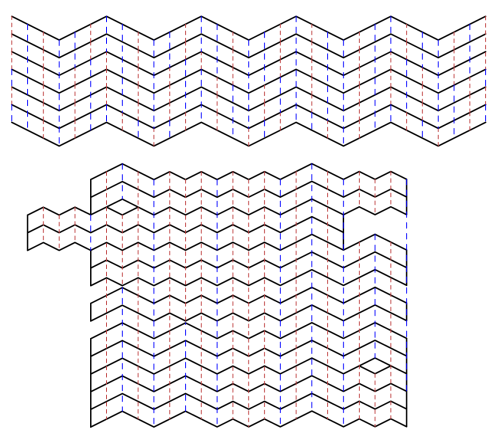
Breaking the Symmetry

We return to our initial definition of quasicrystal spheres and make one modification. Any packing made of the 4 zonohedral unit cells (Figure 1, right) can be further decomposed into packings consisting of only two different rhombohedra (Figure 1, left). Unlike the 4-tile packings however, replacing the zonohedral tiles with only rhombohedra requires breaking the icosahedral symmetry that neatly existed in our earlier definition. Given this modified tiling, we can generate modified quasicrystal sphere polyhedra which can be decomposed into strips using the same method. In this less symmetric version, however, the unrolled strips can no longer be separated into six identical groups. Instead, each edge direction group may be unique.

It appears that in many cases, when we compare the unrolled 2D pattern for a quasicrystal sphere and its corresponding asymmetric modification, we may have nearly the same pattern, the main difference being that the mountain and valley folds change. Figure 3 demonstrates an example for two similar spheres, one with icosahedral symmetry and the other without, each containing three strips per zone group, for a total of 18 strips per sphere. Red and blue lines represent mountain and valley folds, respectively. In this case, the six long central strips for each zone group in both spheres have identical zig-zag cut outlines (in black).

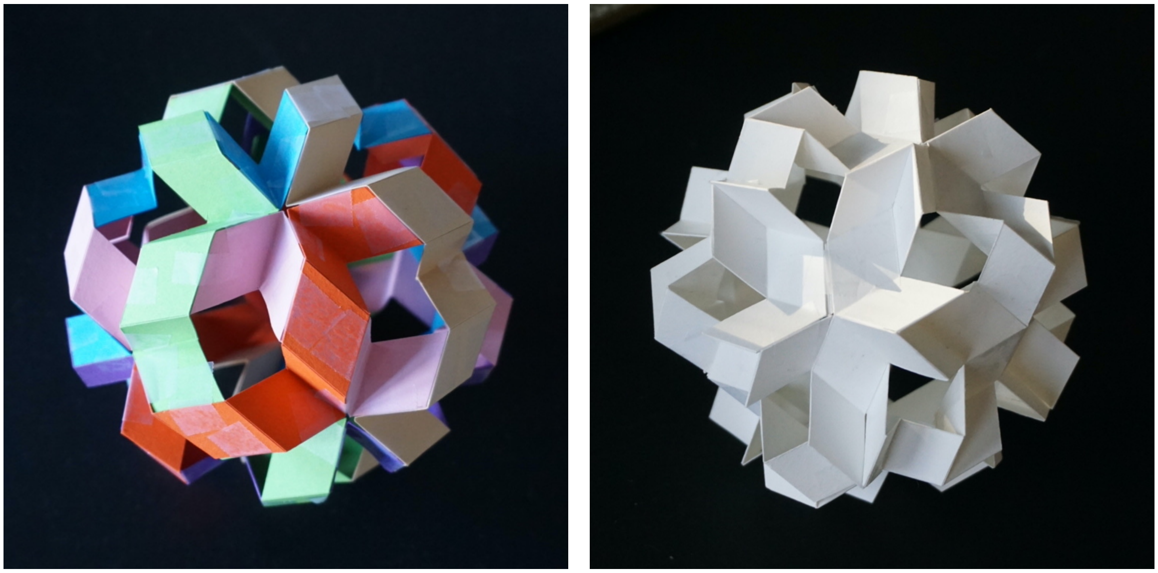


(a)



(b)

Figure 3: *Examples of cut-and-score guide sheets for preparing the paper strips for a pair of similar quasicrystal spheres: (a) icosahedrally symmetric sphere (b) modified asymmetric sphere.*



(a) **(b)**

Figure 4: *Photographs of constructed paper models, each using only the six primary central strips from the top section in Figure 3: (a) icosahedrally symmetric sphere, each band in a different color, (b) modified asymmetric sphere, challenging the viewer to follow the bands without the colors.*

Conclusion

We have demonstrated a new method for building icosahedral quasicrystal sphere structures out of folded paper strips. Only a few simple examples are shown, but the same method can be applied to construct spheres of any size. This study is merely one small observation in the fascinating world of three-dimensional quasicrystals, and the explorations are surely to be continued.

Acknowledgements

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