Middle School Classroom Explorations of Polyhex and Polyomino Fitting Puzzles

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Abstract

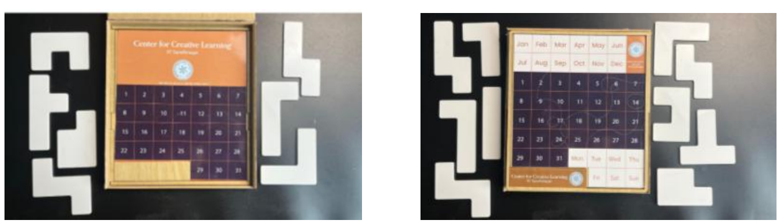
In this paper, we share our experience of geometric and algebraic explorations of the puzzle ‘A-Puzzle-A-Day’ and its variations with middle school students as a part of our weekly online STEM program. The puzzle consists of a set of pieces and a board with calendar dates. The goal is to use the puzzle as a starter activity in the classrooms and eventually lead the class to explore the symmetry and algebra of polyomino and polyhex pieces. As a part of the STEM program, the puzzle is going to 1000+ schools in India along with 80+ other engaging models/puzzles.

Polyhex and Polyomino Fitting Puzzles

A polyomino is a plane geometric figure formed by joining squares edge to edge. Polyomino pieces have been a source of recreation for a long time and numerous polyomino puzzles have served people’s delight. The names for an n-omino have been inspired by domino for two squares, thus giving tromino for three, tetromino for four and so on. A polyhex is a hexagonal cousin of a polyomino with hexagons joined edge to edge, and follows the same name convention with hex as suffix. Amongst the numerous polyomino-fitting puzzles is the ‘A-Puzzle-A-Day’ Puzzle [1]. We have extended the puzzle design with polyhex pieces over a hexagonal grid, Month-Day-Date version with polyomino pieces, and conducted geometric and algebraic explorations over the pieces in our online sessions.

Versions of A-Puzzle-A-Day

The first version (Date Puzzle) consists of 7 unique pentomino pieces and a board with numbers from 1 to 31 as in Figure 1(a) to represent the dates. The task is to cover all the dates using any 6 out of 7 pieces except today’s. Solutions exist for all the dates. The next version (Date-Day-Month) in Figure 1(b), consists of 7 pentomino and 3 tetromino pieces. The board has months (January to December), dates (1 to 31) and days (Monday to Sunday). The task is to cover the board such that all the months, dates and days are hidden except current date-day-month. Considering leap and non-leap years and 7 possibilities of starting day for each, there can be 14 unique calendars.

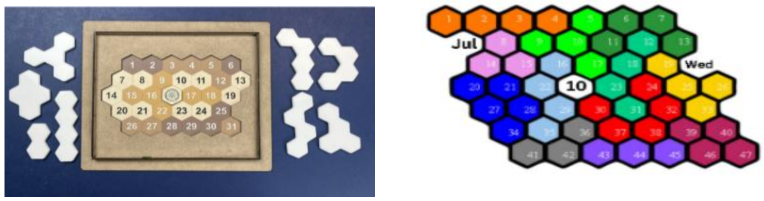


(a) (b)

Figure 1: *Polyomino fitting puzzles (a) Date puzzle with 7 Pentominoes, (b) Date-Month-Day puzzle with 10 Polyomino pieces.*

Polyomino-fitting puzzles are an NP-Hard problem [3]. We have used Dancing Links Algorithm [6], to find the solutions for the given pieces and date-day-month positions. The challenge was to come up with the board design and choice of minimum polyomino pieces.

In the next version, we changed the pieces to polyhexes. The board is a hexagonal grid with 1 to 31 numbers as dates and pieces consisting of 7 tetra-hexes and one di-hex. See Figure 2(a). All the pieces will be used to cover all dates except the current date. The next version (date-day-month) of the puzzle consists of 7 tetrahex pieces, 3 trihex pieces, and 2 pentahex pieces to cover all the months, dates, and days except today's. See Figure 2(b). We have also written an online interactive tool to display the solutions for all dates in both calendar types [7][8].



(a) (b)

Figure 2: *A-Puzzle-A-Day with polyhexes pieces (a) Version 1, (b) Version 2.*

Our Experience with Middle Schools

We are running a large-scale online STEM program [2] in 1000+ government residential schools in India with 100,000+ girl students from marginalised communities. A box containing raw materials for 80+ toys, models, puzzles and experiment kits is delivered to every school. We conduct biweekly live sessions and students join via zoom or YouTube. In the two sessions on 'Fun with Polyominoes' [4][5], we conducted three activities – first with square grid to enumerate the number of unique polyominoes from n squares up to n=5, second to calculate the area and perimeter of those pieces, and third to solve the puzzle for every day of the month.

Activity 1: Enumerating the Number of Unique Polyominoes from n Squares

Using a square grid paper, the task was to figure out the number of unique polyominoes from joining 'n' squares edge to edge. Starting with 2 squares, the uniqueness of the pieces was determined by symmetry between them. If any of the pieces was symmetrical either by a rotation or a flip, only one of the pieces was counted as unique. With 3, 4 and 5 squares, the number of unique polyominoes were enumerated by adding a new square to the collection of polyominoes obtained in the previous step. The exhaustiveness of the number of pieces was discussed and explored by numbering the sides available for joining the squares, putting the new square along the sides, and identifying the unique pieces by eliminating the symmetric ones. See Figure 3.

(a) (b) (c)

Figure 3: *Polyominoes from 'n' squares (a) Placing square along edges of triomino to get tetromino, (b) Identifying unique pieces, (c) Girls with their work.*

Activity 2: Area and Perimeter of Polyominoes

While the area for a particular n-omino remains the same, the objective of this activity was to explore shapes with different perimeters for the same areas, observe patterns in the way perimeters changed and generalise the maximum perimeter for a given n-omino. This was accomplished through appropriate examples and guided poll questions.

Initially, students were asked to observe the area and perimeter of tromino pieces in Figure 4(a) and answer if area was same, perimeter was same or both area and perimeter were same, followed by three tetromino pieces in Figure 4(b) with the same options. Finding them to be the same, they were subsequently asked to hypothesize if the remaining two tetrominoes in Figure 4(c) would follow the pattern. Though area remains the same for all n-ominoes, the perimeter of the square tetromino was found to be different from the previous three, thus proving their hypothesis wrong and highlighting the importance of a counter example or proof in proving a hypothesis.

(a) (b) (c)

Figure 4: *Poll Questions asked in the session (a)Question-1, (b) Question-2, (c) Question-3.*

The next step was to examine patterns in the perimeters. Two interesting observations were listed:

1. The maximum perimeter increases by 2 as we go from an n-omino to the $n+1$-omino
2. All n-ominoes have even perimeters.

Students were asked to observe the change in perimeter when a square was added to an existing n-omino. The added square shares either one or two of its sides with another square. See Figure 5. At this point, students were asked to determine the number of sides that must be shared to get the maximum perimeter. Recognising that the shared sides should be minimum, the maximum perimeter happens when the squares are joined in a straight line. While other arrangements may give the same maximum perimeter, there is no other polyomino that will exceed the perimeter of the straight n-omino. This led to the generalised formula for the maximum perimeter

$$4 \times n - 2 \times (n-1) \text{ i.e. } 2 \times n + 2,$$

where $n-1$ is the number of shared sides for n squares in a straight configuration. Further they were asked to generalize the perimeter for an n-omino, which they calculated to be $4 \times n - 2 \times k$, where k is the number of shared sides in the configuration.

(a) (b)

Figure 5: Sides being shared as two squares are joined edge to edge (a) One edge, (b) Two edges.

Using the generalized formula, the students could notice that the perimeter will always be even because of the difference between two even numbers. This prompted further inquiry as students questioned the reason behind the difference between two odd numbers also being even.

As an extension activity, we asked the students to explore shapes that can be formed by joining triangles from edge to edge, observe the parity of the perimeter and generalise the formula for the maximum perimeter of such shapes. See Figure 6(a).

Activity 3: Solving the A-Puzzle-A-Day

The first versions of the A-Puzzle-A-Day with polyomino and polyhex pieces were solved by the students for all 31 days of the month. Templates for the puzzles were provided. Some students chose to take the print out of the calendar and filled colours for every day of the month as in Figure 6(b) while some cut out cardboard pieces and coloured them as in Figure 6(c). While some schools shared their solution every day for the particular date, some schools solved all the dates within a few days of the session. The puzzle turned out to be an engaging exercise for the students.

(a) (b) (c)

Figure 6: *Student's work (a) Creating shapes using triangles (b) Solving A-Date-A-Day for 31 days (c) A-Date-A-Day with polyhexes.*

Summary and Conclusions

Polyominoes have been enormously popular for puzzles for many years, but they can also be a rich source of mathematical explorations in the middle school classrooms to develop mathematical thinking. The above exercise can also be extended to observe the minimum perimeters and find the patterns in the way perimeters change by plotting them on a graph against n-ominoes. The Center aspires to create such exploratory experiences for schools in India and also works with master trainers of various schools to inspire the joy of learning and exploration in mathematics.

References

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