From Triangles to Pentagons

Richard Nau[1] and Sharol Nau[2]

[1]Departments of Math and CS, Carleton College; rnau@carleton.edu
[2]Northfield, Minnesota; whitecrow@snakedance.org

Abstract

Two distinct pentagons are used in the formation for two series of artworks. One is derived as the limiting form of pentagons extracted from a spiral of triangles. The other is based on the Goldbach Conjecture..

A Spiral of Triangles

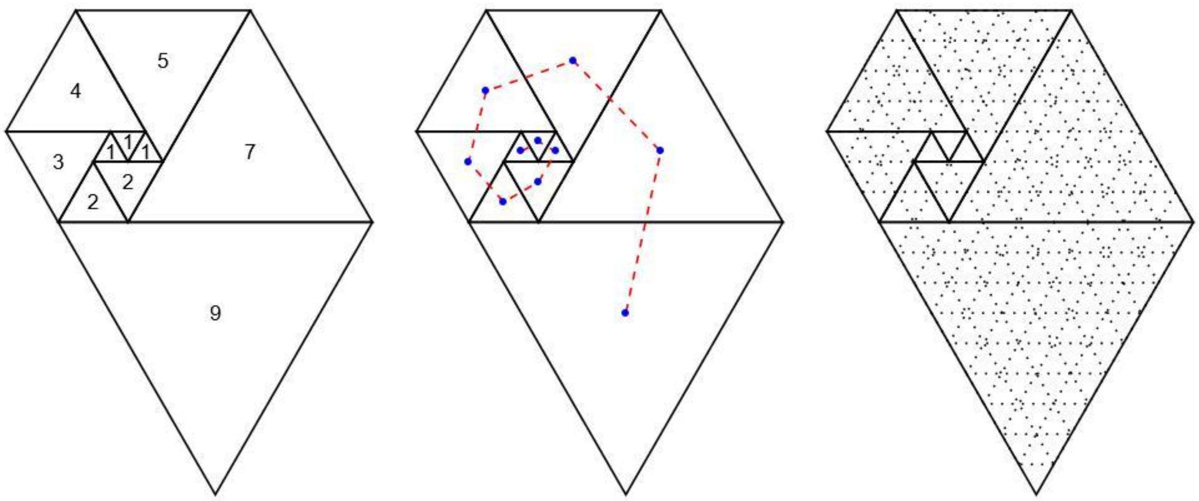


Figure 1: *Spiral of Triangles.*

We begin with a known spiral of triangles. The spiral starts with three *unit triangles*, that is equilateral triangles of side length s = 1, and proceeds clockwise. The fourth triangle, of length s = 2, occurs along the longest side of the original group of three and forms our first pentagon, whose longest side L is also of length 2, as shown in the centers in Figure 1. The next pentagon will have side length L = 3 and is formed by attaching another triangle with s = 2. Placing it on the left leads to a clockwise spiral. A right side placement would yield a counter-clockwise spiral. Next, a triangle with s = 3 makes a pentagon with L = 4. And so on. Some measures for the first few triangles and pentagons are given in Table 1. P(n) is the side length of the nth triangle (beginning with n = 0 for historical reasons); a(n) is its area; A(n) is the accumulated area.

Table 1: *Measures.*

n	0	1	2	3	4	5	6	7	8	9	10
P(n)	1	1	1	2	2	3	4	5	7	9	12
a(n)	1	1	1	4	4	9	16	25	49	81	144
A(n)	1	2	3	7	11	20	36	61	110	191	335

Recurrence Relation for Lengths. After the first few iterations, the matching side lengths, s of the triangles and L of the pentagons, grow according to the recurrence relation

$$P(n) = P(n-2) + P(n-3), \quad \text{for } n = 3,4,5\ldots, \text{ where } P(0) = P(1) = P(2) = 1,$$

yielding the Padovan sequence [3], analogous to the Fibonacci sequence. This relation can be observed in the pentagons in Figure 1 as well as in the second row of Table 1.

The Sequence of Pentagons

As indicated above and shown in Figure 2 below, a sequence of pentagons are embedded in the spiral of triangles. Both the size and shape of the pentagons are of interest.

The Area of the Pentagons. The areas of the successive triangles and pentagons will be expressed in terms of unit triangles. Thus the area of the nth triangle is simply $a(n) = P(n)^2$, as shown in the third row of Table 1 above.

The areas A(n) of the pentagons are summations of the areas a(n) and are given in the fourth row. A closed form for these sums can be observed from the geometry. For example, consider the entire area of the large pentagon in Figure 1. If we add a triangle with s = 5 to the top, a parallelogram is formed; its area is 2 x 9 x 12 = 216. The leading factor of 2 comes from the fact that we're counting unit triangle rather than unit squares. This area less 25, that of the added triangle, agrees with the 191 in Table 1. This generalizes, and can be proved by induction to give

$$A(n) = 2\,P(n)\,P(n+1) - P(n-2)^2, \quad \text{for } n = 3,4,5,\ldots\,.$$

This formula for the sum of the squares of the first n+1 Padovan numbers is equivalent to the one in OEIS [2,4].

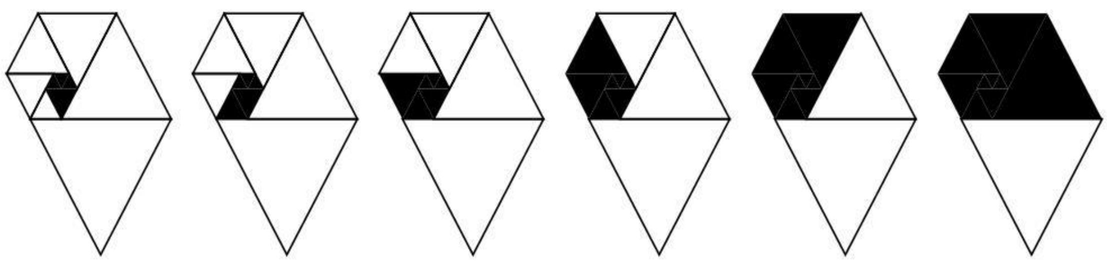


Figure 2: *Evolving Shapes.*

The Shape of the Pentagons. The shapes of the successive pentagons are highlighted in Figure 2, which suggests that there might be a limiting shape, with the same angles, as n grows without bound. To try to control for the growth in area, let's first consider the values of the ratios of successive sides, that is P(n+1) / P(n). As n grows, this ratio approaches a number approximately equal to 1.3247. This limit is not entirely surprising. It can also be obtained from the recurrence relation for the Padovan numbers, whose corresponding characteristic equation is $x^3 = x + 1$. The only real root of this is known as ρ (rho), sometimes referred to as the Plastic Ratio.

To be able to compare shapes we'll normalize by dividing by some lengths. First, if we normalize the size of the limiting pentagon by dividing by its shortest side. Because of the common side ratio, the five sides in the limit must be $1, \rho, \rho^2, \rho^3, \rho^4$. Now, scale it and each of the first six pentagons by dividing their side lengths by the length of their perimeter. We can see the convergent of their shapes in Figure 3, which compares each of the first six pentagons to the limiting one. By the sixth pentagon the difference between it and the limit is barely visible. Figure 4a shows the relative sizes of the second through fifth pentagons. Figure 4b gives a tiling using the limiting shape. That shape needs a name. Plastic Pentagon? Rho Pentagon?

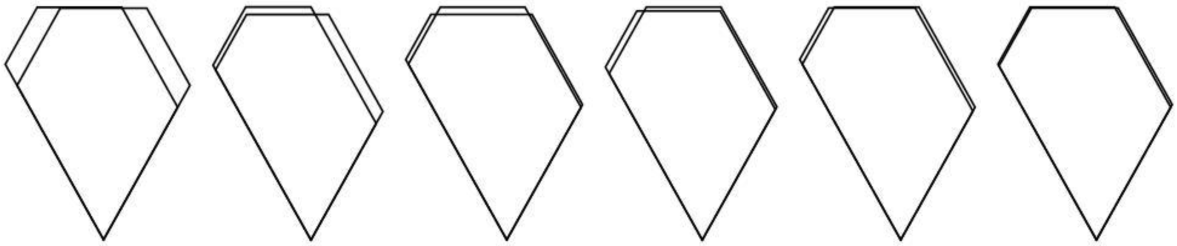
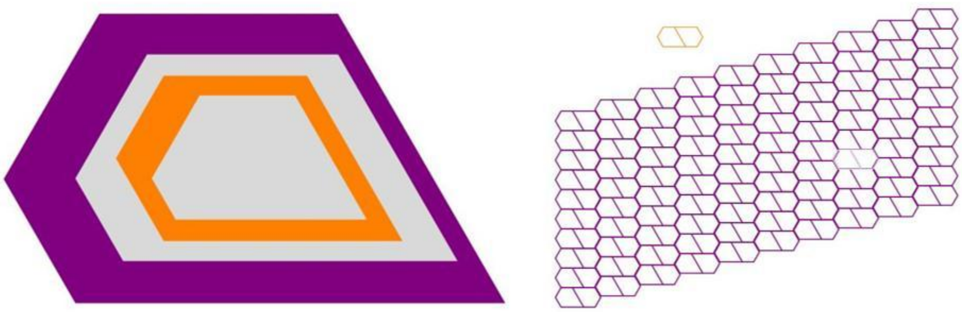


Figure 3: *Convergence of Shapes.*



(a) (b)

Figure 4: *Plastic Pair: (a) Sizes, (b) Tiling.*

The Goldbach Pentagon

The Goldbach Pentagon is similarly based on triangles, in this case isosceles right triangles rather than equilateral triangles. The famous unsolved Goldbach Conjecture claims that any even integer greater than two can be partitioned into the sum of two primes. Consider the grid in Figure 5a. It has 32 = 2×4×4 triangles as each of the 16 squares can be split into 2 such triangles. To achieve the desired balance between shape and space, 13 triangles were used for the shape and the remaining 19 for space.

(a) (b)

Figure 5: *The Goldbach Pentagon: (a) basic pentagon, (b) mosaic.*

Figure 6: *Goldbach Tiling*

An early mosaic pattern by the second author using this shape is shown in the right panel of Figure 5b. Figure 6 shows a larger painting (about 3' by 5') . Although it's not as obvious, the partitioning of 32 into 13 plus 19 was used there as well. For more see [1]

Summary and Conclusions

Two different approaches were taken in converting triangles to pentagons. Beginning with a spiral of equilateral triangles we uncovered a sequence of pentagons which converge to a limiting shape in which the constant rho played an essential role. Also a formula for the sum of squares of the Padovan numbers was found. Beginning with a particular instance of the Goldbach partitioning, a series of paintings in which basic triangles were formed into pentagons and then into more elaborate shapes.

Acknowledgements

Although unbeknownst to them, two organizers of these conferences should be thanked for their recent stimulation. Within the past year we joined the Facebook group monitored by one of them on tilings and tessellations. It was in his work that we first saw the spiral of triangles. Just a few months later we attended a symposium at Carleton College at which another organizer gave an invited talk on tessellations renewing and updating our interest in that topic. Thanks to the helpful reviewers and to Carleton for funding.

References

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