Changing the Topology of Polyominoids Through Rigid Origami

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Abstract

Arrange a collection of identical squares side-by-side to form a connected geometric figure — this is a polyomino. Given a 2D polyomino with holes, can we fold it into a 3D polyominoid with a desirable topology? For instance, can all holes be eliminated, resulting in a surface that deformation retracts to a point, or can we transform it into a cylinder or a sphere? We seek to achieve such transformations purely through rigid folding, without tearing the material or overlapping squares. In this paper, we interweave the study of polyominoes and polyominoids with techniques resembling those from origami and kirigami to introduce a mathematical model for classifying and manipulating these transformations. Finally, we explore potential applications in product and puzzle design: we pitch several toy ideas and provide examples of innovative lamp structures where topology plays an interesting role in shaping the properties and effects of light.

Introduction

A *polyomino* is a collection of squares in the two dimensional regular square lattice with a connected surface interior. A *polyominoid* is a collection of squares in the three dimensional cubical lattice with a connected surface interior. A *folding pattern* is an assignment of "mountain", "valley", and "flat" folds to the interior edges of a polyomino that instructs them to fold precisely 90°, -90°, and 0°, respectively, to obtain a polyominoid. A folding pattern on a polyomino is *valid* if it does not force any two squares to overlap during the folding process and does not require any cuts along its interior edges. Further, the folding must be *rigid*; that is, it must preserve planar faces, only allowing bending of the material along specified creases. In this work, we explore folding patterns on polyominoes with intriguing topologies to achieve specific design goals. A collection of animations of these folding processes can be found at [4].

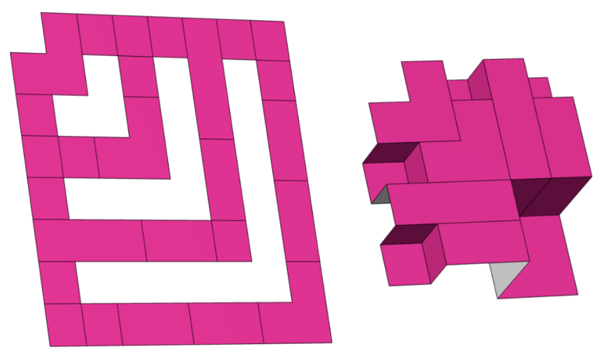

Figure 1: *A polyomino with holes that can be folded into a polyominoid that is homeomorphic to a disk.*

For instance, one of our goals is to find folding patterns that *close* holes in polyominoes (a *hole* in a polyomino is a finite connected component of its complement) and produce polyominoids that are homeomorphic to a disk, as shown in Figure 1. We also give some examples of obstructions that prevent polyominoes from being folded into polyominoids with trivial topology. Finally, from studying folding patterns on small polyominoes, e.g., with a single hole, we describe a technique that we call *compatible patching* that allows us to create tilings which ensure the polyominoid generated by our expanded folding pattern inherits desirable topological properties.

Closing Simple Holes

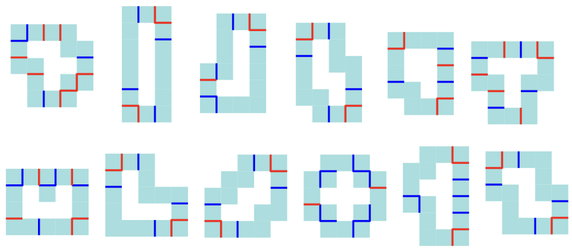

Figure 2: *All pentomino fenestrations, each with a closing marked.*

We say that a hole in a polyomino P is *closable* if there exists a valid folding pattern on P such that in the resulting polyominoid all the inner perimeter edges of P have been identified with at least one other edge in the inner perimeter. A *closing folding pattern*, or *closing* for short, is a folding pattern that closes all holes in a polyomino.

A *fenestration* is a polyomino containing exactly one hole and all squares that share a zero- or one-dimensional face with it. A *square fenestration* is a fenestration containing an $n \times n$ square shaped hole. This terminology is inspired by the object's visual likeness to windows in a building.

In Figure 2, we provide an example of a closing for each of the 12 fenestrations that have their hole shaped like a pentomino. On visual diagrams, we use red and blue lines to denote mountains and valleys, respectively, and no mark for flat. We invite the reader to find different closing folding patterns on these 12 polyominoes. We have also checked that fenestrations with holes shaped as monomino, dominoes, trominoes, and tetrominoes are closable. When we get to hexominoes or larger, this remains an open problem. In [1] it is proven that rigid foldability is NP-hard which suggests this is not a trivial pursuit even for fenestrations with small holes.

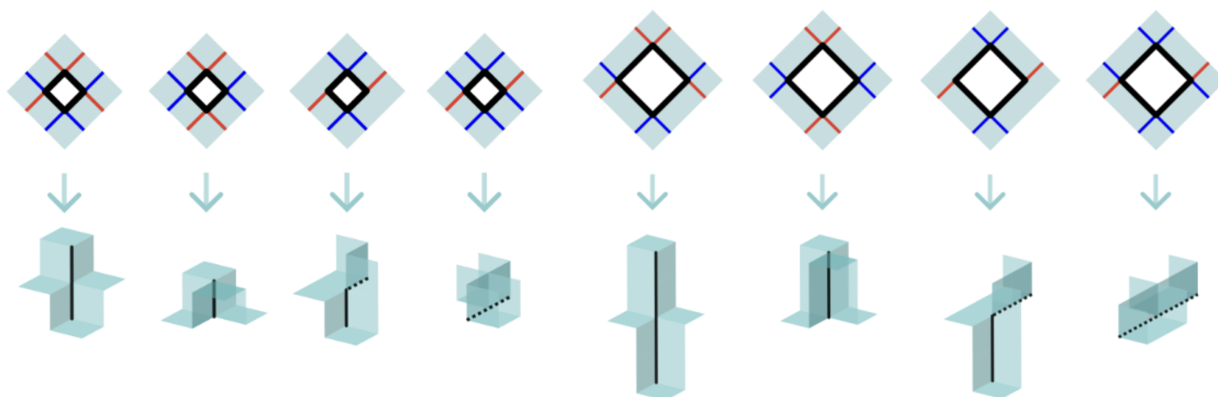

Figure 3: *The four polyominoids generated by closing a square fenestration (sizes* 1×1 *and* 2×2 *shown).*

We claim that there exist exactly 4 closings for every $n \times n$ square fenestration for all $n \in \mathbb{N}$. By exhaustive search, we have checked that this is true for the first three elements of the sequence, and present the first two in Figure 3. We count folding patterns up to isometries of $\mathbb{Z}^2$, meaning a pattern, and thus the resulting polyominoid, is not considered distinct after reflection, rotation, or color-swapping.

Conjecture: There exist exactly 4 closings for every square fenestration.

Two key steps we foresee as essential for proving this conjecture are:

1. To facilitate a closing of a square fenestration, mountain and valley folds must be assigned to either 6 or 8 of the interior edges incident to its four corner squares.
2. Exactly 2 possible gluings of inner-perimeter edges exist, one in which all 4 sides of the square come together, and another where two pairs of perpendicular sides are glued.

Patching and Tiling

Let A and B be polyominoes, a 1D-*patching* of A and B is an identification (or gluing) of a subset of their edges (one-dimensional faces). A 2D-*patching* of A and B is an identification (gluing) of a subset of their square two-dimensional faces. We construct a *compatible patching* by choosing a pair of folding patterns, one on A and one on B that agree on their intersection (the faces that have been identified).

In Figure 4 (left), we see a closable compatible 1D-*patching*, where the folding patterns on A and B agree along the edge where they are glued, exactly one mountain and one flat fold are perfectly aligned.

Figure 4: *On the left, a closable* 1*D-patching of the* 1×1 *and* 2×2 *square fenestrations. We highlight the folding patterns that are compatible. On the right, A subset of a closable periodic tiling, formed by an iteration of* 1*D-patching, with a folding homeomorphic to a disk.*

In Figure 5 (left), we exhibit a closable 2D-*patching* where the two polyominoes are overlapped and their intersection contains an identically assigned mountain and valley fold.

One can build a non-closable patching by chosing a subpolyomino A' with two or more squares that must lay flat (all interior edges have neither mountain nor valley assigned) in every closing of A. Select a subpolyomino B' that must be folded in every closing of B, and patch A and B by gluing A' and B' to generate a non-closable polyomino.

In Figure 5 (right), we give an example of a non-closable 2D-*patching*. We know the resulting polyomino from the patching is non-closable because we have enumerated all closings of $n \times n$ square fenestrations for $n \leq 3$. For the particular patching that we give in Figure 5, there do not exist any folding patterns on the 1×1 and 3×3 square fenestrations that would result in a compatible folding pattern that is a closing on the resulting polyomino.

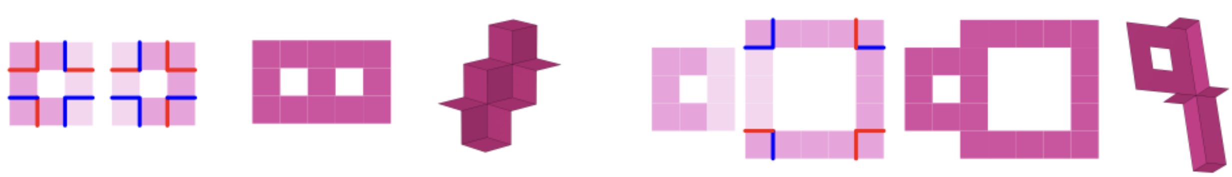

Figure 5: *On the left, a closable* 2*D-patching of two* 1×1 *square fenestrations. We highlight the folding patterns that are compatible. On the right, a non-closable polyomino built from 2D-patching a* 1×1 *and a* 3×3 *square fenestration.*

One can build a closable periodic tiling by selecting a fenestration A with a folding pattern f and two translation vectors v and w such that Av and A are compatibly patched, Aw and A are compatibly patched, and patching $A + Av + Aw + Avw$ does not create new holes.

In Figure 4 (right), an example of a closable periodic tiling is constructed by iteratively patching with the 2×1 fenestration (by translating it with the $(3, 1)$ and $(4, 0)$ vectors).

Product and Puzzle Design

It is clear that folding patterns on certain polyominoes can generate a disk, a cylinder, a sphere, a torus, or a two-holed torus. For torsion to appear, one needs to look at embedding hyperbolic square polyforms in Tesseract and higher dimensional cubes or cubical tesselations [2]. Foldings that purposefully leave some fenestrations open (reminiscent of a punctured-sphere topology) may generate polyominoids particularly well suited for design and engineering applications such as lampshades, as they can let light through their remaining apertures; see Figure 6 for two examples.

Folding polyominoes into polyominoids has the potential to generate a multitude of beautiful and useful objects which are portable and dynamically responsive to changing needs. For instance, for solar engineering or camping gear, which are well-known practical uses of rigid origami. On the recreational side, foldable polyominoes could be designed as a modular toy that encourages building structures, much like Lego® or Magnetiles®. We also noticed that some disk polyominoids share a similarity to the pixelated terrains of Minecraft®. A fun puzzle can be generated by printing a terrain image on a polyomino and prompting one to find the unique folding which realizes a specified polyominoid, such as shown in Figure 6. The printed image serves as a hint in identifying desired orientations, foldings, and gluings of edges.

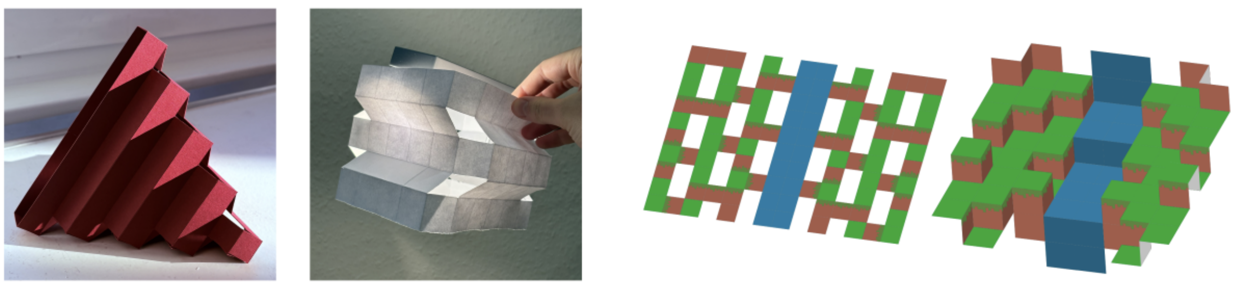

Figure 6: *Two cylindrical lamps built from folding patterns on polyominoes. An example of a folding puzzle. The goal is to fold the polyomino into the target polyominoid.*

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