Études in Symmetric Pattern Drawing

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Abstract

Symetrical patterns, planar ornaments and rosettes form a math/art activity among all ages and cultures. This article explores problems and possibilities that arise in the context of stylus driven symmetry drawing apps. Besides highlightling several visualisation challenges, we will also present suggestions for creative art/math activities on the borderline of creativity and structure.

Ornaments in Math, in Real Life and on Computers

The term *ornament* has different meanings in architecture, arts, handicraft, mathematics or even music. What they all have in common is a reference to something decorative, possibly in a rather abstract than concrete way. For our purposes, we imagine an ornament as a symmetric pattern drawn in the plane. We will mainly focus on ornaments that arise from the classical 17 planar crystallographic groups but we also deal with patterns that exhibit one center of rotation (mandala type ornaments), patterns in the hyperbolic plane, patterns on the sphere and several more.

My personal connection to ornaments, their mathematical modelling and algorithmic realization dates back to 1986 when I wrote one of the first ornament drawing programs (at that time on an Atari 1040ST) for a big symmetry exhibition in Darmstadt, Germany. This computer was one of the first machines that offered a mouse as a kind of quasi-analog input device. People were able to draw strokes with the mouse and the computer repeated them according to the rules of the chosen symmetry. Already at that time, with its limited technical possibilities, the program was a magnet to visitors. They often spent long periods immersing themselves in a blend of strict rules and creativity. Since then, I have developed several incarnations of this general idea with growing technical possibilities but also growing demand for expressiveness and artistic freedom. They are used in various exhibitions as well as internet scenarios and they inspired others to create similar software. Currently, the most sophisticated incarnation is the symmetry drawing app iOrnament [6] that makes heavy use of GPU (graphics processing unit) technology and is headed towards pen driven artistic expressiveness. This article attempts to illuminate the project from three perspectives. First, some of the mathematical and implementation challenges of such a project are exemplified from a developer perspective. Second, we consider the perspective of the artist seeking for mathematics-related artistic expressiveness. The third perspective is that of the math or art educator who wants to provide mathematical or artistic challenges for others in the context of the app.

As described above, we define an ornament as an image that extends infinitely across the Euclidean plane and exhibits symmetries. To model this mathematically, we also call a function $f\colon \mathbb{R}^2 \to F$ an *ornament* where F is a suitably chosen *color space*, for instance $\{0,1\}$ for black and white images, $[0,1]$ for grayscale, or $[0,1]^3$ for RGB color. A *symmetry* of an ornament f is an isometry S of the plane such that $f(x) = f(S(x))$ for all $x \in \mathbb{R}^2$. In the Euclidean plane, there are four types of isometries: the *translations*, *rotations*, *reflections*, and *glide reflections*. If f is an ornament, then the set of symmetries $\mathcal{S} := \{S \text{ isometry} \mid f(x) = f(S(x))\ \forall x \in \mathbb{R}^2\}$ forms a group, the *symmetry group* of the ornament [1]. Note that every ornament f has the identity as a trivial symmetry. We restrict ourselves to so-called *discrete* symmetry groups in which no orbit of a point has an accumulation point. Only few types of possible symmetry

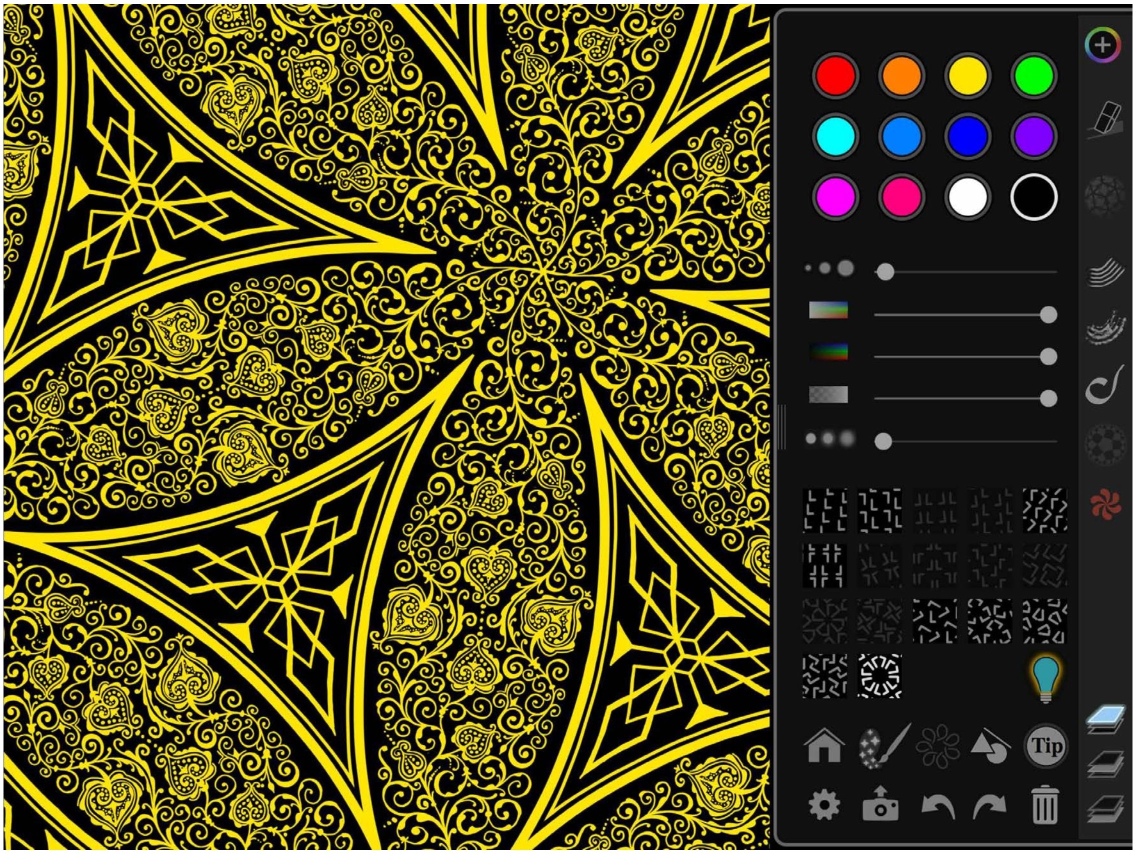


Figure 1: *The UI of the app and an ornament created by London-based calligrapher Seb Lester.*

groups remain in this case: Depending on their translational symmetries, ornaments can be classified into *rosettes* (no translational symmetries), *frieze patterns* (translational symmetries in one direction) or *wallpaper ornaments* (translational symmetries in two linearly independent directions). While there are infinitely many types of rosettes, differing in the order of the center of rotation, there are exactly 7 types of frieze patterns and 17 different wallpaper groups.

As a consequence of the two directions of translational symmetries, every wallpaper ornament contains a compact region – the *fundamental cell* – that can be used as a stencil to fill the plane by the action of the symmetry group [1]. In a certain sense, wallpaper ornaments are a hermaphrodit between finiteness and infinity. They reach out infinitely but the region of artistic freedom is limited to a finite region. Perhaps this is the reason why M. C. Escher dealt intensively with planar ornaments: he tried to capture various aspects of infinity in many of his artistic works.

"Just" Drawing a Line

Based on the above considerations, the natural starting point for an ornament drawing program is the selection of a specific type of symmetry. Then, strokes or images created by the user will be repeated according to the rules of the selected symmetry. To state it in poetic terms: Drawing in an ornament drawing program is like playing a flute in a room with an incredible echo. You play something and it comes back from all directions.

Ideally, a symmetry drawing program offers a workflow that enables the user to react to their own creations, use lucky accidents, and quickly produce unexpected and unpredictable results. Designing a good ornament drawing program depends largely on such non-mathematical requirements. iOrnament strives to be a tool with a natural "look and feel" throughout the entire user experience, offering a wide range of artistic expressiveness that empowers both artists and non-artists in their creations. A gallery of user creations is shown in Figure 2.

It might sound surprising but already supplying the code for drawing a simple line with a stylus carries its specific challenges: The resolution within the fundamental cell should be high, while also supporting strong zoom-in and zoom-out capabilities to create a sense of infinity. Potentially, this results in very sparse or very dense sample points for stylus positions as they have to be handled over a huge range of drawing speeds. Therefore, a fast rendering and interpolation algorithm is essential. To save rendering time, all strokes are mapped to a single copy of the fundamental cell and drawn there. After that, the ornament is generated by creating copies of the fundamental cell according to the symmetry group using fast GPU techniques. However, there are still situations, e.g., having local sub-rosettes in a mandala, where a certain stroke has to be rendered over 1000 times in real time.

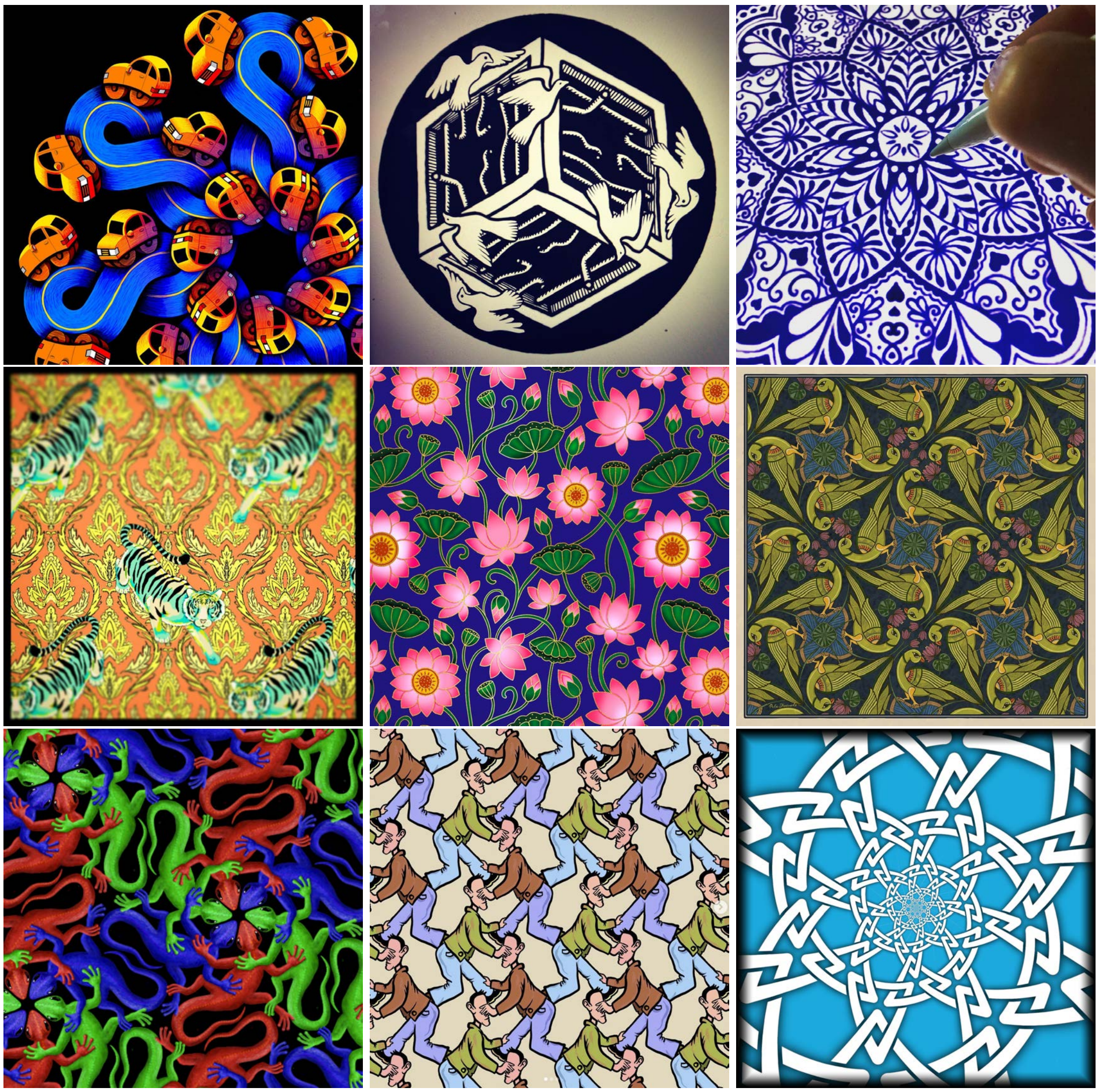
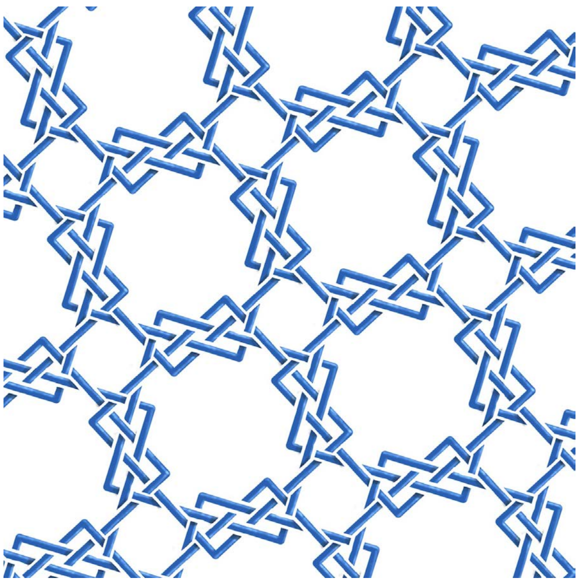


Figure 2: *A gallery of images created by users, together with links to more of their artwork. In order of the grid: Miriam Martincic, Luca Vallese, Rike Brakenbusch, Amber Ansleym, Prita Dhaimade x2, Ulli Haag, Antonio Lirio and myself. Links are provided in the reference section.*

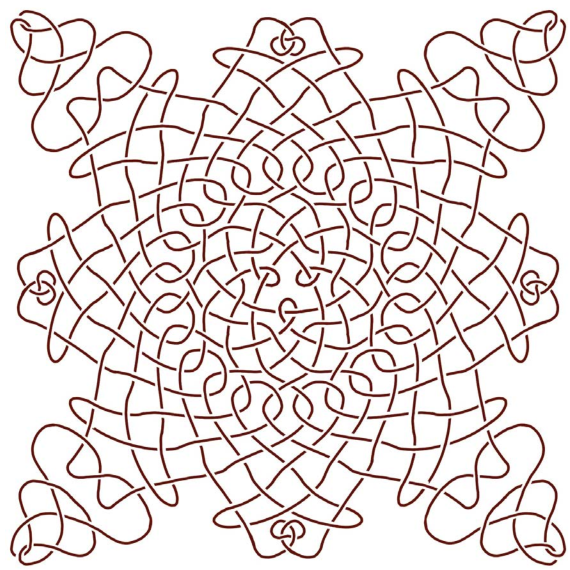
Symmetry Breaking, Braids, and Knots à la Dürer

In iOrnament, it is possible to switch between symmetry types that share the same fundamental cell shape during the drawing process. This opens up very intriguing and challenging mathematical, artistic and educational options for designing ornaments.

For example, one can switch from the symmetry group $p4m$ (generated by a 4-fold rotation, a 2-fold rotation, and a reflection along the connecting axis) to $p4$ (the corresponding group without reflection), thereby intentionally breaking symmetries of $p4m$. A particularly appealing exercise is to draw one single line segment whose start and end points coincide under $p4m$ (resulting in a symmetric pattern of closed paths). By then switching to $p4$ and selectively erasing parts, these paths can be turned into braided or woven structures as in Figure 3a. An interesting problem is to classify the types of ornaments that arise in this way.



(a) *A braid pattern. Similar structures appear in many Islamic star patterns.*



(b) *Rosette pattern with interwoven lines, created using the symmetry breaking technique.*

Figure 3: *Overlaying symmetries.*

A second application of this symmetry breaking technique involves rosette symmetries. Here, one can create ornaments of interwoven lines that exhibit different rotational orders in different regions. For instance, a rosette can have a 3-fold symmetry at the center while having a 4-fold symmetry near the boundary. Appealing closed loops can be achieved by ensuring that adjacent regions differ by a prime factor in rotational order. The ornament in Figure 3b follows the sequence $4 \to 8 \to 24 \to 12 \to 6 \to 3$ with the paths again transformed into woven structures. The overall pattern has no non-trivial global symmetry but is composed of highly symmetric local elements. Comparable techniques were used by Albrecht Dürer in the creation of his famous knot drawings.

Hyperbolic and Spherical Symmetries

The iOrnament software offers extensive possibilities to explore symmetry groups beyond those of the Euclidean plane. A world analogous to the Euclidean ornament theory can be developed in planes of constant positive curvature, i.e., the surface of a sphere, and constant negative curvature, i.e., the hyperbolic plane. In many cases, the graphical content of an ornament can be transferred from the Euclidean plane to other

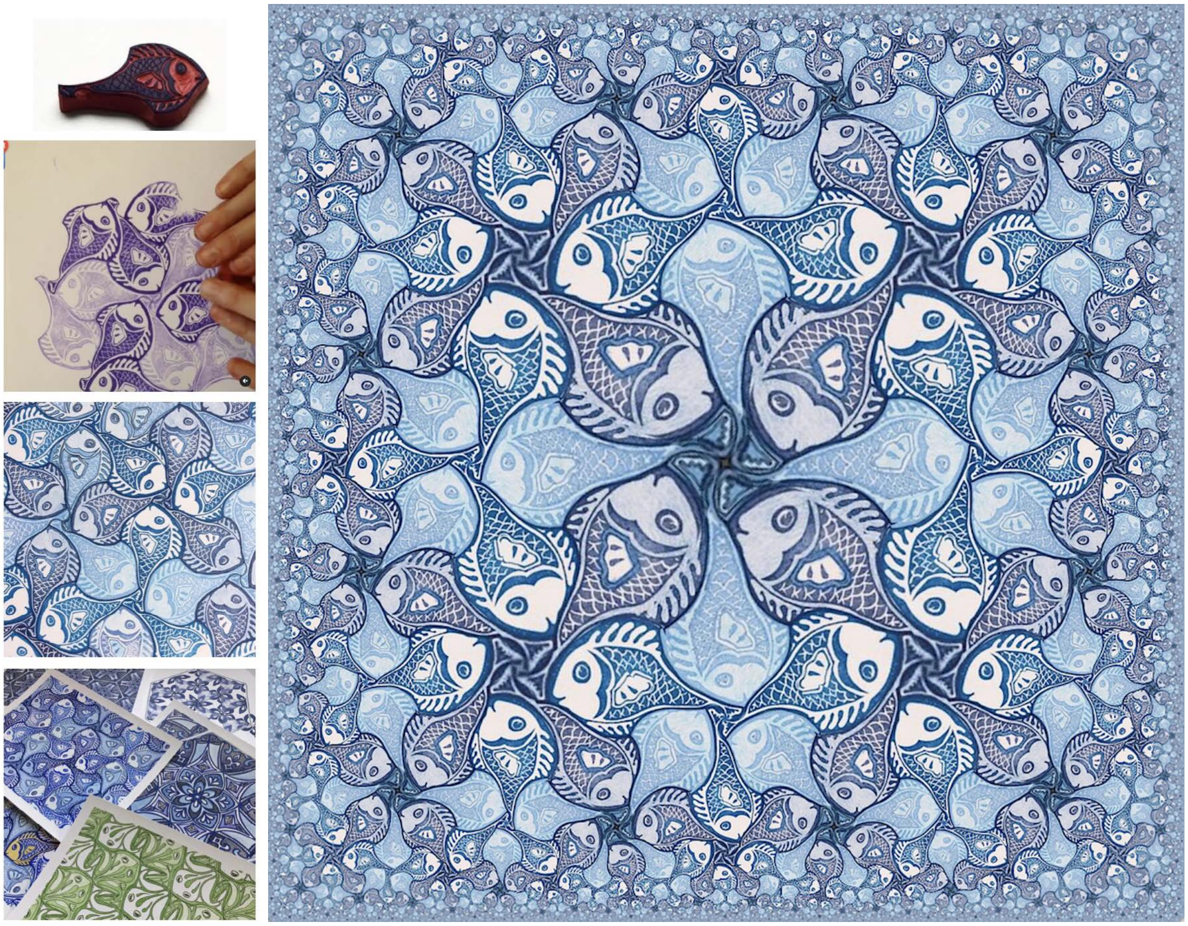


Figure 4: *Some creations make a full cycle from digital to real and back to digital. In the above fish pattern based on work of Marta Harvey [3], the original design was created in iOrnament. The fundamental fish was then exported and carved in soft foam. From that, a colored print using three shades of blue was created. The print was reimported into Cinderella (using the copy trace tool), transformed into a hyperbolic ornament and finally conformally transformed to a square.*

geometries. The theory here is diverse and to some extent a bit subtle. This shall be illustrated with a relatively simple example in hyperbolic geometry.

Consider a *kaleidoscopic pattern*, that is, an ornament whose fundamental cell is a triangle and whose symmetry group is generated by reflections only. To produce a closed ornament, the individual angles must be integer divisors of 180°. In the Euclidean plane, the angle sum of such a triangle is always 180°. So, for example, a Euclidean kaleidoscope triangle may be given by corner angles of 90°, 60°, and 30°. If the angles were changed so that their sum becomes smaller or larger than 180°, the triangle can no longer be realized in the Euclidean plane. However, as long as all angles are still integer divisors of 180°, it can be realized as a triangle in the hyperbolic plane or on the sphere and a reflection group arises which can serve as the basis of an ornament. On the sphere, this leads to the symmetries underlying the Platonic solids. In addition to these finitely many types of symmetry, there are two infinite classes of spherical frieze and rosette ornaments. In the hyperbolic plane, the structures are even richer: since the angle sum of a triangle can be made arbitrarily small, centers of rotation of arbitrarily high order appear, resulting in an infinite and rich class of symmetry groups. For a complete classification, see Conway, Burgiel, and Goodman-Strauss [1].

A special feature of iOrnament is its ability to transfer the artistic content of the ornament – that is, the content of a fundamental cell – from one geometry to another with minimal visible distortion, using conformal mapping. One of the most fundamental results in complex analysis, the Riemann mapping theorem, states that any region bounded by a Jordan curve can be mapped conformally onto the interior of a disk. Applying the theorem in both directions makes it possible to map the interior of a triangle in Euclidean geometry (including its artistic content) onto a corresponding triangle in hyperbolic geometry. Such Riemann mappings are unique up to a Möbius transformation with three degrees of freedom, which provides just enough flexibility to map the corners of the triangles onto each other. Once the combinatorics of the symmetry group are fixed, the conformal transfer of the ornament is essentially unique [2]. Filling the hyperbolic plane can then be achieved by the Schwarz reflection principle.

Conceptually, this approach is simple. However, it requires considerable mathematical trickery to perform it in practice, in particular under real time constraints. In iOrnament, a gradient flow method is used to compute an equilibrium on a spring network, creating a harmonic map. To deal with the boundary conditions, the Schwarz reflection principle is applied to virtually extend the equilibrium map consistently beyond the borders of the fundamental cell. The algorithm is implemented directly on the GPU, providing a smooth and visually pleasing transition between the symmetry groups. See [5] for details on the algorithm.

Special Effects

Over the roughly 12 years since the first release of iOrnament, various requests from users have emerged – among them, a growing demand for advanced pens that create special effects, such as gold and glitter. Many of these effects require sophisticated rendering techniques, often involving GPU programming. Without going into detail here, Figure 5 highlights several of these special effects, each representing a distinct mathematical or programming challenge.

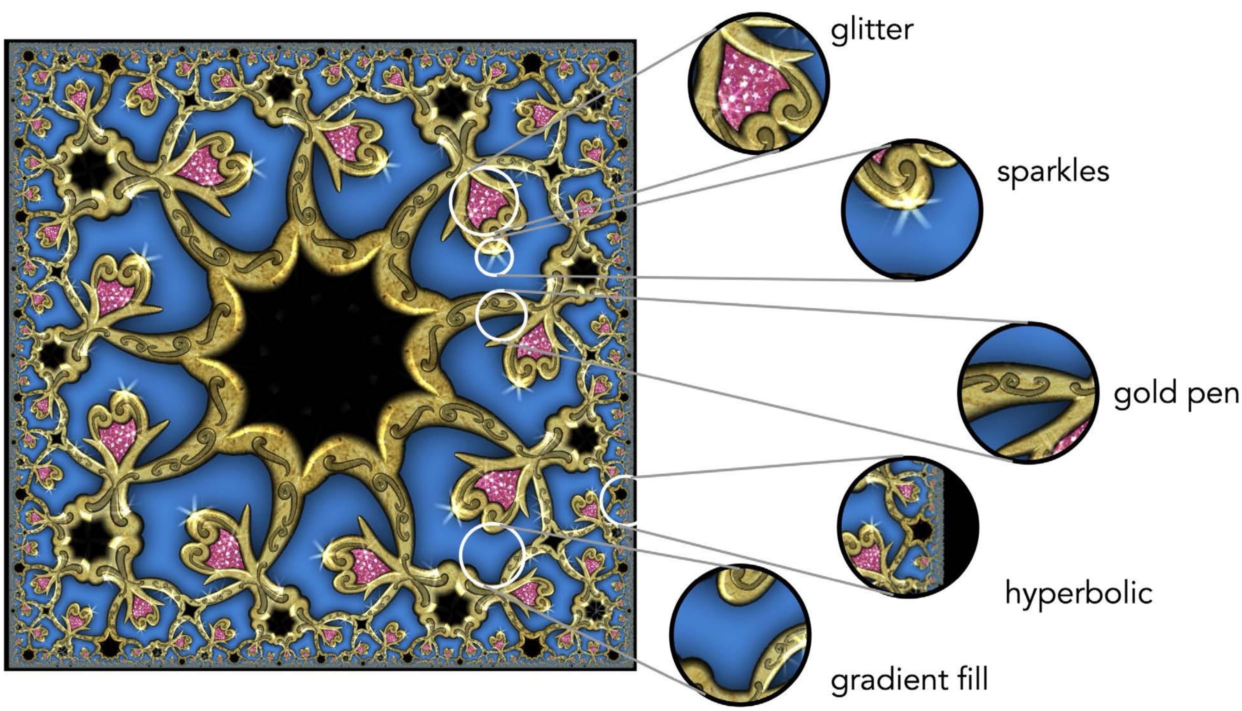


Figure 5: *Various challenges related to the implementation of special effects.*

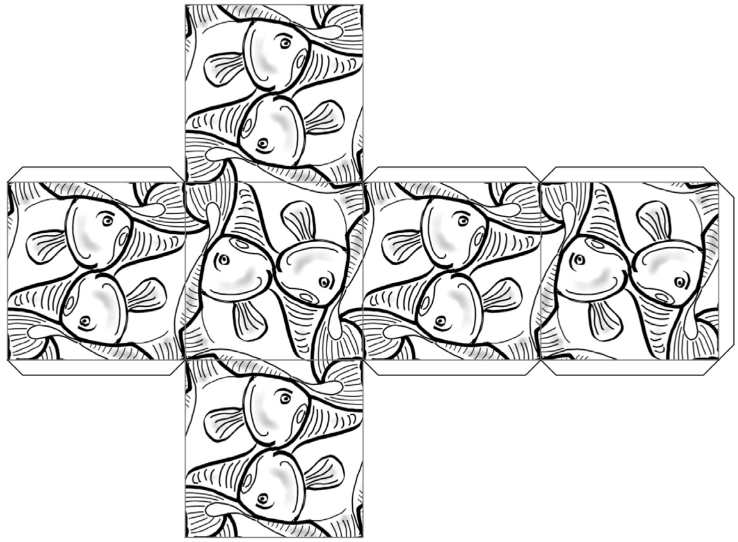
Export to the Real World

There are numerous examples where users created patterns in iOrnament and then transformed them into real world artwork. The calligrapher and font designer Seb Lester (www.seblester.com) is well known for his mix of calligraphic penmanship and the use of digital tools. He has worked intensively with iOrnament and some of his prints are produced using hybrid techniques where the core of the drawing was generated in iOrnament and further elements were added using other methods. One of my personal favourites is his giant letter "S". It is impossible to reproduce it in a way that does the original print justice. Figure 6 shows a small copy of the overall creation along with a zoom into one of the details. The overall design is based on a circle packing approach, combined with calligraphic knot work. The small circles contain references to ornamental patterns, ranging from ancient cultures, via Celtic knots, Baroque ornaments to Art Nouveau, modern company logos and emoji. The major part of the body letter "S" was created in iOrnament. The creation made extensive use of a local symmetry feature that allows the user to place local rosette ornaments as a substructure of a larger composition.

Figure 6: *Seb Lester's "S" [4] incorporates ornamental pattern design through ages and cultures.*

A more general concept of real-world export from iOrnament is the following: The companion app iOrnament Crafter [7] can be used to create 3D paper models from a user's ornamental creations. This workflow (*create your own ornament* ⟶ *transform into a 3D symmetry* ⟶ *create 3D paper models*) can be used for interesting educational activities. Mathematically, this feature makes use of the transformation of Euclidean symmetry groups to spherical ones, which are often based on the symmetries of the Platonic solids. Flattening the situation onto the faces of the corresponding Platonic solid results in a surface on which the ornamental creation can reside and for which a flat craft sheet exists. Figure 7a gives an impression how we applied this approach during an outreach activity on our university campus. This activity can be extended by deliberately creating black-and-white printouts that are hand-colored afterwards. Interesting mathematical challenges can arise in that context. For instance, one might ask for the minimal number of colors needed to color the fish in the cubical map in Figure 7b such that adjacent fish get different colors.

(a) *Workshop impressions.*



(b) *The net of fishy cube.*

Figure 7: *Possible outreach activities involving ornaments.*

Conclusion

This article attempts to provide an impression of the various activities and challenges that arise at the intersection of mathematics, computer science, and art in the context of drawing symmetric patterns. Many topics have only been touched upon; each could easily fill an article of similar length. Overall, it can be said that the computer-supported creation of ornaments resonates with both scientists and artists, as well as with the general public.

References

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Links to artists in the sampler: Miriam Martincic: https://www.miriamdraws.com, Luca Vallese https://www.zentequerente.com, Rike Brakenbusch https://rikebrakebusch.de, Amber Ansley https://www.instagram.com/scorpi_oh/, Prita Dhaimade https://www.behance.net/pritadhaimade, Antonio Lirio https://www.instagram.com/toni_lirio/, iOrnament Instagram https://www.instagram.com/iornament/.