On the Hunt for Flexible Polyhedra

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Abstract

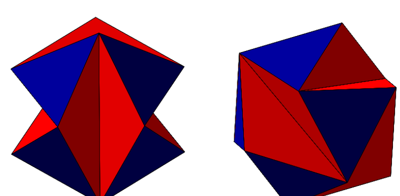
A polyhedron in Euclidean 3-space is called infinitesimally flexible if there exist tangent vectors of motions of the vertices that retain edge lengths, but change distances of at least one non-connected pair of vertices. In this paper, we describe a construction of such polyhedra inspired by examining models built from plastic triangles. We illustrate a method to construct polyhedra by gluing strips at their boundaries. Following the presented approach, we provide an example of a polyhedron that is indeed infinitesimally flexible.

Introduction

Polyhedra are geometrical objects that arise in various scientific contexts and are appealing to a wide range of researchers. Over the years, many notions of polyhedra have been established in the literature [6, 7]. Here, a polyhedron is a 3-dimensional object whose surface consists of vertices, edges and planar polygonal facets. Renowned artists such as M. C. Escher have incorporated these objects into numerous artworks, creating paintings that continue to fascinate scientific and non-scientific audiences to this day [8].

Our research focuses on polyhedra that exhibit intriguing geometric properties, such as convexity, symmetry and rigidity. One class of polyhedra that is of particular interest to us is the class of (infinitesimally) flexible polyhedra. These are polyhedra that exhibit (infinitesimal) motions of the vertices that preserve the shape (congruence type) of each face, but change the distance between at least one pair of unconnected vertices [15]. Infinitesimal motions can be thought of as an assignment of velocity vectors to the vertices of a polyhedron that do not change the edge lengths. Kaleidocycles are examples of flexible polyhedra [11]. Schattschneider and Walker [13] present various Kaleidocycles that are decorated with tesselations designed by M. C. Escher.

(a)



(b)

Figure 1: *(a) Different flex states of a 3D-printed Kaleidocycle, (b) Different views of Jessen's icosahedron*

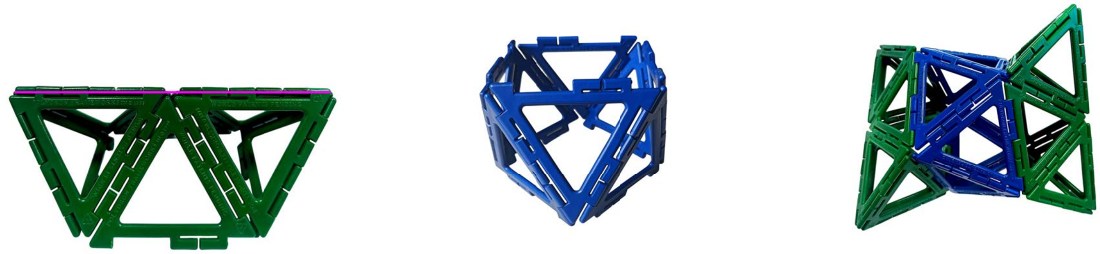
Another well-known example of a flexible polyhedron is the Bricard octahedron which has been constructed by Bricard [4]. This polyhedron is flexible and combinatorially isomorphic to the octahedron, i.e. the Platonic solid consisting of 8 equilateral triangles as facets. We refer the reader to [5, 9, 14] for more studies on flexible polyhedra. Furthermore, there exist various examples of polyhedra that are infinitesimally flexible yet not flexible. For instance, Goldberg [10] states that Jessen's orthogonal icosahedron is infinitesimally flexible, but does not admit a finite flex. In his work, Goldberg presents further examples of such polyhedra and refers to them as 'shaky'.

The significance of infinitesimally flexible motions comes from real-world applications. When constructing infinitesimally flexible polyhedra out of material such as steel or plastic, the structure exhibits instability, although being mathematically rigid. This is due to imperfections in the material and finite stiffness, allowing bending and deformation of a constructed polyhedron. The absence of infinitesimally flexible motions is a sufficient condition for rigidity, so finding infinitesimally flexible polyhedra is a first step to finding flexible polyhedra.

In this paper, we explore a construction method for polyhedra that is based on combining different strips (see the following section) with the aim of finding flexible polyhedra. We utilise physical models built from plastic triangles from the company Polydron [12] to observe possible (infinitesimal) motions of the corresponding polyhedra. Hence, the polyhedra corresponding to physical models with an apparent flex form promising candidates for flexible polyhedra. We then employ a Julia-program [3] to determine whether the constructed polyhedra are indeed flexible. In particular, we make use of the Julia package `GeoCombSurfX` [1], which has been designed to explore polyhedra and assemblies of polyhedra.

Construction

We formulate our construction method based on the physical models made out of plastic triangles. For this construction, we introduce two key ingredients, namely strips and caps. A *strip* is a set of at least six triangles connected along their edges such that each triangle is adjacent to exactly two other triangles. We assume the strips in this paper to be orientable. Hence, the *boundary edges*, i.e. the edges of the triangles of a strip that are not incident to any other triangle, form exactly two edge-loops. By arbitrarily choosing one of those loops and assigning it the label "bottom", each triangle is either directed "upward" (u), if it shares an edge with the bottom loop, or "downward" (d), if it does not. Thus, we can describe a strip by a cycle with entries u and d describing the triangle orientations in order. In particular, the length of a cycle is the number of triangles a strip contains. We call the strip with corresponding cycle $(u, d, u, d, \ldots, u, d)$ of length $2n \geq 6$, $n \in \mathbb{N}$ the *standard strip of length* $2n$. Furthermore, a *cap* is obtained from a strip with an even number of edges on one of its boundaries, by glueing (identifying) a starting pair of adjacent edges on one of the boundaries and then iteratively identifying their neighbouring edges. Thus, the boundary loop is transformed into a path of non-boundary edges (illustrated in pink in Figure 2a). Once a starting pair of adjacent boundary edges is chosen, the resulting cap is uniquely determined. We mark the starting pair used to form a cap in the cycle representation of the given strip by underlining the corresponding triangles. For example, the cap obtained from the cycle $(\underline{u}, d, \underline{u}, d, u, d, u, d)$ (see Figure 2a) is constructed from the standard strip of length 8 (Figure 2b) and has exactly one boundary with 4 edges.



(a) **(b)** **(c)**

Figure 2: *Physical model construction. In (a), the pink line shows the path formed by glueing the boundary.*

Since strips and caps exhibit many degrees of freedom, it is very challenging to parametrise their vertex locations fully by means of computer algebra systems. This is why it is helpful to employ triangles made from plastic to gain intuition and insights into complex geometries formed by glueing together such strips. Finally, we describe our desired construction method. In particular, we construct models corresponding to polyhedra by glueing together strips and caps along their boundaries in a sequence such that the bottom boundary of one strip/cap is compatible with the top boundary of the next strip/cap. For instance, the model illustrated in Figure 2c can be constructed from glueing the cap shown in Figure 2a onto the standard strip of length 8 (see Figure 2b) and then closing the resulting boundary by glueing it onto another cap as illustrated in Figure 2a. The model shown in Figure 2c can therefore be described by the sequence

$$((\underline{u},d,\underline{u},d,u,d,u,d),\ (d,u,d,u,d,u,d,u),\ (\underline{d},u,\underline{d},u,d,u,d,u)).$$

An Infinitesimally Flexible Polyhedron

Next, we present an example of an infinitesimally flexible polyhedron that results from the construction method described above. This polyhedron consists of 40 vertices, 114 edges and 76 faces. The physical model describing the polyhedron can be constructed from two caps consisting of 8 triangles each and five strips consisting of 12 triangles each, see Figure 3. The caps (built with blue triangles) used for the above construction can both be described by the sequence $(d,d,d,\underline{u},d,d,d,\underline{u})$. Further, the strips assembled using yellow and green triangles can all be represented by the sequences $(d,d,d,u,u,u,d,d,d,u,u,u)$ and (d,u,d,u,d,u,d,u,d,u), respectively.

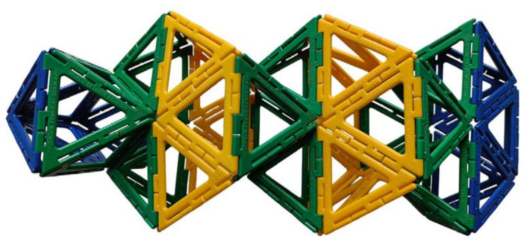


Figure 3: *Plastic model of an infinitesimally flexible polyhedron. The different colours of the triangles indicate the caps and strips used for this construction.*

By constructing the physical model of this polyhedron from plastic triangles, we have observed that this model exhibits some flexible motions. We illustrate the model at different flex states in Figure 4.

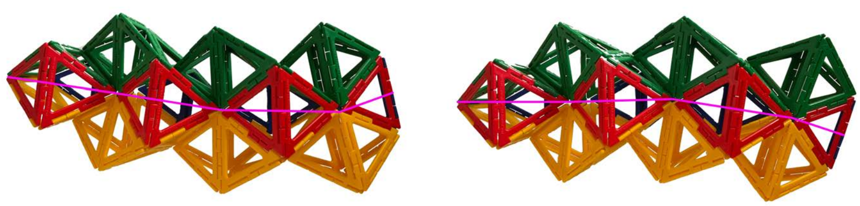


Figure 4: *The purple line indicates the flex. The different colours of the triangles emphasise the flex.*

By encoding the illustrated model into an object that can be analysed with the package `GeoCombSurfX` in Julia, we have been able to verify that the arising polyhedron is indeed infinitesimally flexible. Further, by employing a particle simulation [2], we have shown that this polyhedron is not flexible. Hence, the flexes exhibited by the physical model result from the infinitesimal flexibility of the polyhedron together with the imperfections in the plastic pieces and their finite stiffness. Additionally, we have not found any examples of rigid polyhedra whose corresponding plastic models suggested a finite flex. This indicates that the triangles are stiff enough not to allow deformations of actually rigid candidates.

Conclusion

In this paper, we have presented a method to generate polyhedra by glueing strips and caps along their boundaries. In our experiment, we have observed that the polyhedra that result from the presented construction method have a high tendency to be infinitesimally flexible. In future work, we aim to study this construction method and its capabilities in more detail. So far, the polyhedra that arise from our method are either not infinitesimally flexible or infinitesimally flexible but not flexible. That means we have not been able to combine strips and caps to construct a polyhedron that is indeed flexible. Thus, our hunt for a flexible polyhedron still continues.

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