Spiky Soccer Balls: Generalized Polar Zonohedral Clusters

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Abstract

We extend Webster's method for generating polar zonohedral clusters by removing the requirement that the seed polyhedron be circumscribable. Applying this construction to Kleetopes of the dodecahedron, we produce spiky forms that can be faceted by connecting adjacent zonohedral poles to produce truncations of the icosahedron, resulting in various soccer ball-like forms. We analyze the relationship between truncation, pyramid height, and edge length ratios, providing an explicit formula linking these parameters. These considerations inspired our art submission featuring generalized polar zonohedral domes forming a standard truncated icosahedron, with the spiky form appearing as negative space within the clusters.

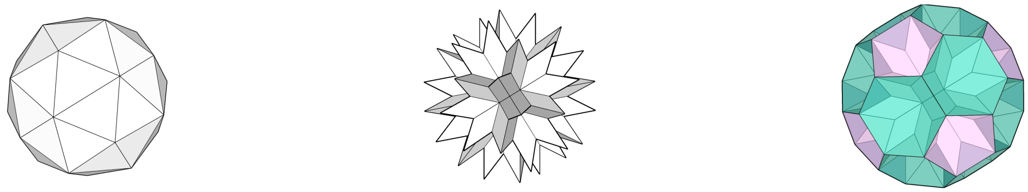
Introduction

This work builds on Webster's study of polar zonohedral clusters [4], which investigates how polar zonohedra can be arranged to locally fill space around a point by associating one to each face of a circumscribable polyhedron with regular faces. Webster examines the star figure formed by connecting the centre O to each vertex V_i of a regular face, producing equal-length line segments. This star figure defines a polar zonohedron as the Minkowski sum of these line segments [1], with a central axis extending from O through the centroid C of the face and perpendicular to it. Applying this construction to all the faces of the seed polyhedron produces a cluster of polar zonohedra (see [4], Figure 4). These clusters inherit the symmetries of their seed polyhedra and fill space around the central point without gaps or overlaps. Webster classifies such clusters derived from Platonic solids, Archimedean solids, prisms, antiprisms, and circumscribable Johnson solids.

We extend Webster's method by removing the circumscribability requirement, applying it to a broader class of polyhedra. If the seed polyhedron is not circumscribable, the line segments $\overrightarrow{OV_i}$ vary in length. However, the resulting generalized polar zonohedral clusters retain parallelogram faces and a polar axis extending from O through the centroid C of a face (though not necessarily perpendicular to it). We then facet this zonohedral cluster by connecting adjacent poles of the spiky form, producing a structure related to the dual of the seed polyhedron. This process can be repeated indefinitely, each time using the faceted form as the new seed polyhedron.

We apply this construction to non-circumscribable polyhedra, focusing on Kleetopes of the dodecahedron. Recall that a Kleetope is a type of augmentation formed by placing a pyramid on each face of a polyhedron. Our method is as follows: start with a Kleetope of the dodecahedron and apply the construction to generate a spiky form (or cluster). This form consists of 60 repeated parallelepipeds arranged around a common central point. Faceting this spiky form by connecting adjacent poles produces a variation of the truncated icosahedron, with edge lengths determined by the distances between the centroids of adjacent faces of the Kleetope. While this does not yield the standard dual of the Kleetope, it produces a polyhedron that is combinatorially equivalent. Each variation consists of 12 regular pentagonal faces with edge length A and 20 irregular hexagonal faces with two distinct edge lengths, A and B. These forms represent different truncations of the icosahedron, resulting in various soccer ball-like forms.

To compare these forms, we calculate the ratio B/A, which equals 1 for the standard $1/3$ truncation of the icosahedron. In general, the truncation value t ranges from 0 (no truncation) to $1/2$ (maximum truncation,



(a) **(b)** **(c)**

Figure 1: *(a) Pentakis Dodecahedron, (b) Spiky Form, (c) Truncated Icosahedron ($B/A \approx 0.887$).*

leading to the Archimedean icosidodecahedron). The corresponding Kleetope of the dodecahedron (assumed to have edge length 1) has pyramid height h, related to the ratio B/A and truncation t by:

$$0 \leq \frac{1-2t}{t} = \frac{B}{A} = \frac{2}{5}\sqrt{5\left(5-2\sqrt{5}\right)}h + \frac{5+\sqrt{5}}{10} \approx 0.6498h + 0.7236$$

for $h \geq -\frac{\phi^2}{2\sqrt{3-\phi}} \approx -1.114$ (maximum truncation $1/2$), where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Allowing negative pyramid heights enables inversion of the pyramid augmentation on dodecahedral faces, creating excavated dodecahedra.

We apply this construction to three Kleetopes of the dodecahedron, producing spiky clusters whose faceted forms correspond to truncations of the icosahedron. These examples inspired our art submission, featuring generalized polar zonohedral domes or caps forming a standard truncated icosahedron.

Pentakis Dodecahedron Spiky

Our first example applies this method to the dual of the truncated icosahedron, the pentakis dodecahedron (see Figure 1a). The resulting spiky form is shown in Figure 1b. Since the pentakis dodecahedron is a geodesic polyhedron made entirely of triangular faces, the generalized polar zonohedral clusters take the form of simple parallelepipeds. This structure consists of 180 parallelogram faces, divided into two types: rhombic faces in the pentagonal florets and a second type of parallelogram in the hexagonal florets. It can also be decomposed into 60 oblique rhombic prisms.

The faceted spiky form closely resembles the original Archimedean structure, but it does not correspond to the standard $1/3$ truncation of the icosahedron. Instead, it gives a ratio of

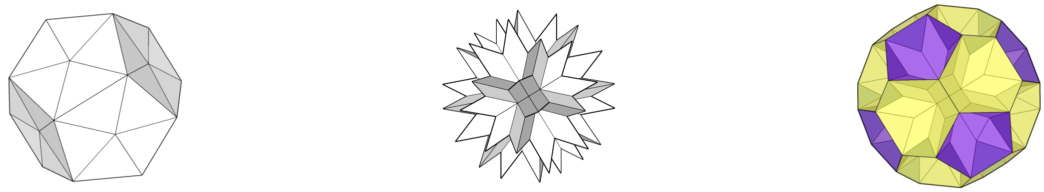
$$\frac{B}{A} = \frac{27+3\sqrt{5}}{38} \approx 0.887$$

(see Figure 1c), resulting in slightly irregular hexagons. The truncation value in this case is $\frac{38}{103+3\sqrt{5}} \approx 0.34637$.

Excavated Dodecahedron Spiky

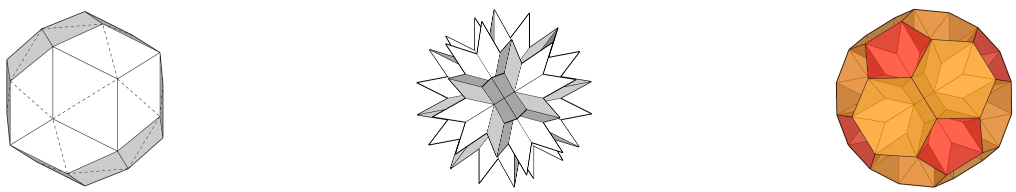
The biscribed truncated icosahedron (see Figure 2c) is an example of a more spherical soccer ball, studied in [2]. This form has the notable property that all its faces are equidistant from the centre while also being circumscribable. As a result, it creates a more spherical structure with improved ball-handling and flight characteristics, which influenced Nike's Geo series ball designs (see [3] for a thorough history of evolving soccer ball designs and their connections to art and math). The biscribed truncated icosahedron has a edge-length ratio given by

$$\frac{1}{2}\left(\sqrt{\frac{3}{5}\left(5+2\sqrt{5}\right)}-1\right) \approx 0.692.$$



(a) **(b)** **(c)**

Figure 2: *(a) Excavated Dodecahedron, (b) Spiky Form, (c) Truncated Icosahedron ($B/A \approx 0.692$).*



(a) **(b)** **(c)**

Figure 3: *(a) Rhombic Triaconahedron, (b) Spiky Form, (c) Truncated Icosahedron ($B/A = 1$).*

It corresponds to a $\frac{10}{15+\sqrt{75+30\sqrt{5}}} \approx 0.37147$ truncation of the icosahedron and a seed Kleetope of the dodecahedron with height:

$$h = \frac{\sqrt{5} - \sqrt{75 + 30\sqrt{5}} + 10}{\left(-5 + \sqrt{5}\right)\sqrt{10 - 2\sqrt{5}}} \approx -0.04866600895$$

This forms a concave, excavated dodecahedron (Figure 2a), with the corresponding spiky form shown in Figure 2b, generating the (biscribed) truncated icosahedron in Figure 2c.

Triangulated Rhombic Triaconahedron Spiky

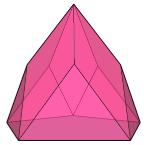
In order to achieve the Archimedean truncated icosahedron with $B/A = 1$ and a $1/3$ truncation, we require a Kleetope of the dodecahedron, which is also a triangulation of the rhombic triacontahedron (see Figure 3a, where dashed lines indicate coplanar faces). The resulting spiky form, shown in Figure 3b, can also be generated by the 162$^{\text{nd}}$ stellation (of 227) of the rhombic triacontahedron. In this case, the faceted spiky form corresponds to the Archimedean truncated icosahedron, where both the pentagonal and hexagonal faces are regular (see Figure 3c).

Art Submission

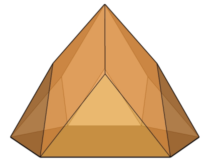
In each example, the spiky structure appears as the negative space within 32 generalized polar zonohedral domes, joined along their triangular faces. These domes are generated from a star of line segments extending from the centre of the seed Kleetope of the dodecahedron to rings of vertices, with alternating degrees of $5, 6, 5, 6, 5, 6$ for hexagonal domes and $6, 6, 6, 6, 6$ for pentagonal domes (Figures 4a and 4b).

Our sculpture (Figure 5) embodies this structure, featuring 20 hexagonal and 12 pentagonal generalized polar zonohedral domes, crafted from coloured transparent acrylic. Each dome is manually cut, scored, folded, and secured with tape to form a 6.25-inch diameter Archimedean truncated icosahedron.

The transparent surface reveals the spiky negative space, emphasizing three key elements: the faces, the generalized polar zonohedra, and the interior void. Analogous colours create a harmonious blend, enhancing



(a)



(b)

Figure 4: *Generalized Polar Zonohedral Domes: (a) Pentagonal, (b) Hexagonal.*

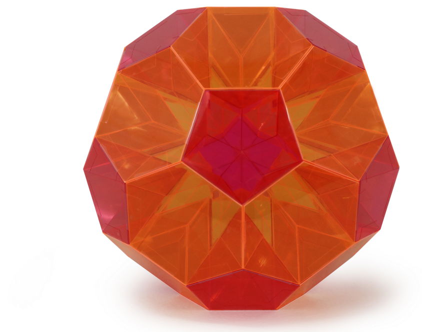


Figure 5: *Spiky Soccer*

the interplay of surface form and space. Sculpturally, the spiky interior emerges as a defining aspect of the composition.

Summary and Conclusions

In this work, we extended Webster's method for generating polar zonohedral clusters by removing the circumscribability constraint. Applying this to Kleetopes of the dodecahedron, we created spiky clusters whose faceted forms correspond to truncations of the icosahedron, exploring soccer ball-like structures and relationships between truncation, pyramid height, and edge length ratios. These constructs inspired an art submission featuring generalized polar zonohedral domes forming a truncated icosahedron, with the spiky form as negative space.

Future work will explore additional seed polyhedra and further explore the duality between zonohedral cluster formation and faceting. Integrating these constructions with computational tools could also facilitate the generation of complex forms for both mathematical and artistic purposes.

References

[1] G. Hart. "The Joy of Polar Zonohedra." *Bridges Conference Proceedings*, Online, Aug. 1–3, 2021, pp. 7–14. http://archive.bridgesmathart.org/2021/bridges2021-7.html.

[2] P. Huybers. "In Search of the Roundest Soccer Ball." *Int. Conference on Adaptability in Design and Construction*, Eindhoven, The Netherlands, July 3-5, 2006, pp. 6.115–6.121. https://www.irbnet.de/daten/iconda/CIB10895.pdf.

[3] D. Swart. "Soccer Ball Symmetry." *Bridges Conference Proceedings*, Baltimore, Maryland, USA, July 29 - Aug. 1, 2015, pp. 151–158. https://archive.bridgesmathart.org/2015/bridges2015-151.pdf.

[4] P. Webster. "Polar Zonohedral Helices and Clusters." *Bridges Conference Proceedings*, Halifax, Nova Scotia, Canada, July 27-31, 2023, pp. 329–336. https://archive.bridgesmathart.org/2023/bridges2023-329.pdf.