Creating Art by Constructing Poncelet Grids and Elliptic Billiards

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Abstract

This article illustrates the construction of a grid within an ellipse based on the theory of mathematical billiards which can then be given an ornamental design.

Introduction

The mathematical billiard problem studies the motion of a mass within a closed region that reflects off the boundary following the basic physical principle whereas the angle of incidence is equal to the angle of reflection. An excellent survey of the theory and results can be found in [2].When a closed orbit is found, reliance on Poncelet's Porism allows for the construction of a grid within specific domains. The caustic of such orbits is defined as the curve that remains tangent to each segment of the orbit. Examples of elliptic and hyperbolic caustics are discussed.

Construction

A simple example of such motion is illustrated below in Figure 1 with a mass constrained to lie within a circle which reflects off the boundary following the basic reflection principle. Given that triangle AOB is isosceles the incident angle at B will equal the reflected angle at A, and so on. The central angle at O will then equal 180° –2•∠BAO. If an integer multiple of this central angle equals a multiple of 360°, then the orbit will eventually close. Otherwise, the path will continue indefinitely. In the example below, the curve closes after 3 segments.

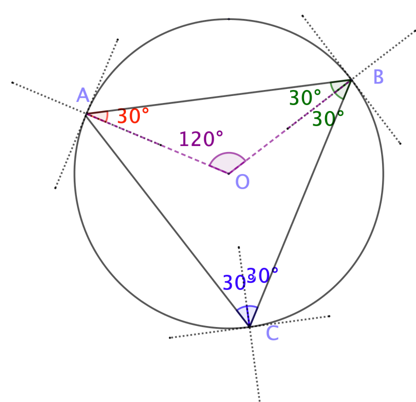
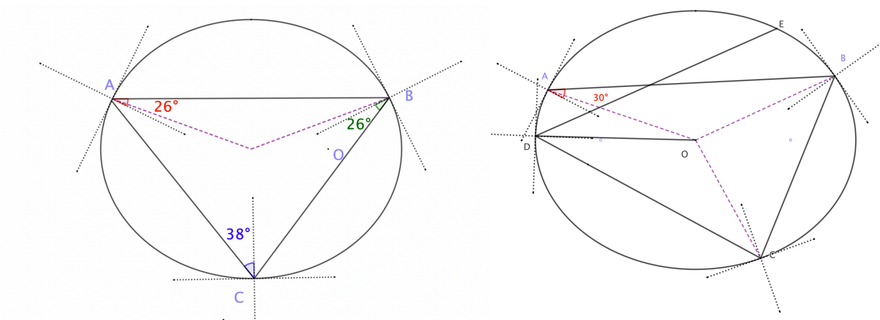


Figure 1: Illustration of a closed path within a circle.

The problem is more complicated for an elliptic boundary, as the distance from the center changes. Figure 2 illustrates two types of solutions. Shown in Figure 2a below is an example of an orbit that closes after three segments (which can be found numerically), but with just a slight change in the initial angle, as shown in Figure 2b, the path can continue indefinitely, though illustrated is merely the lack of closure after 4 segments. Within this paper closed paths are based on achieving a desired level of tolerance.

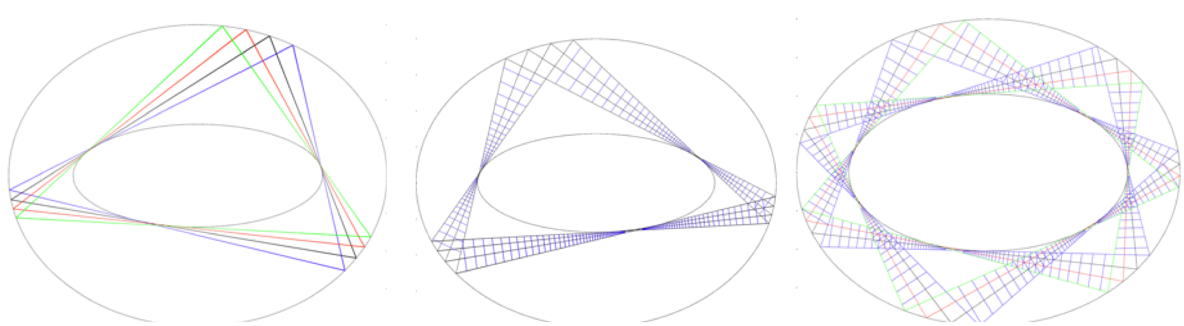


(a) (b)

Figure 2: Illustration of paths within an ellipse (a) closed path after 3 segments (b) not closed

In the case where the orbit is closed, a grid can then be constructed following the Poncelet Porism. A thorough analysis of the Porism is provided in [1] which can be summarized by saying that if a polygon can be found that is both inscribed in one conic and circumscribed about another conic, then there are infinitely many such polygons with the same number of sides with this property. This paper only discusses configurations derived from elliptic conics. Illustrated below in Figure 3 is an example of this porism. If a closed path is found, then all successive paths can be determined by changing the starting point and requiring the new initial segment to be tangent to the inscribed ellipse. This interior ellipse, while theoretically guaranteed requires some numerical analysis to ascertain. In Figure 3(a) we see four such polygons (triangles) that are both inscribed and circumscribed about a pair of confocal ellipses, followed by a grid in Figure 3(b) which is obtained by adding line segments perpendicular to the sides of a chosen triangle. The spacing of these perpendiculars, while arbitrary, is chosen so that the interior quadrilaterals are close to square. Grids based on different spacings can aid in the construction of different patterns. Figure 3(c) is an example of an inscribed polygon with 7 sides. Note that the path in Figure3c winds around the center two times before completion. A regular 7-sided polygon solution exists as well.

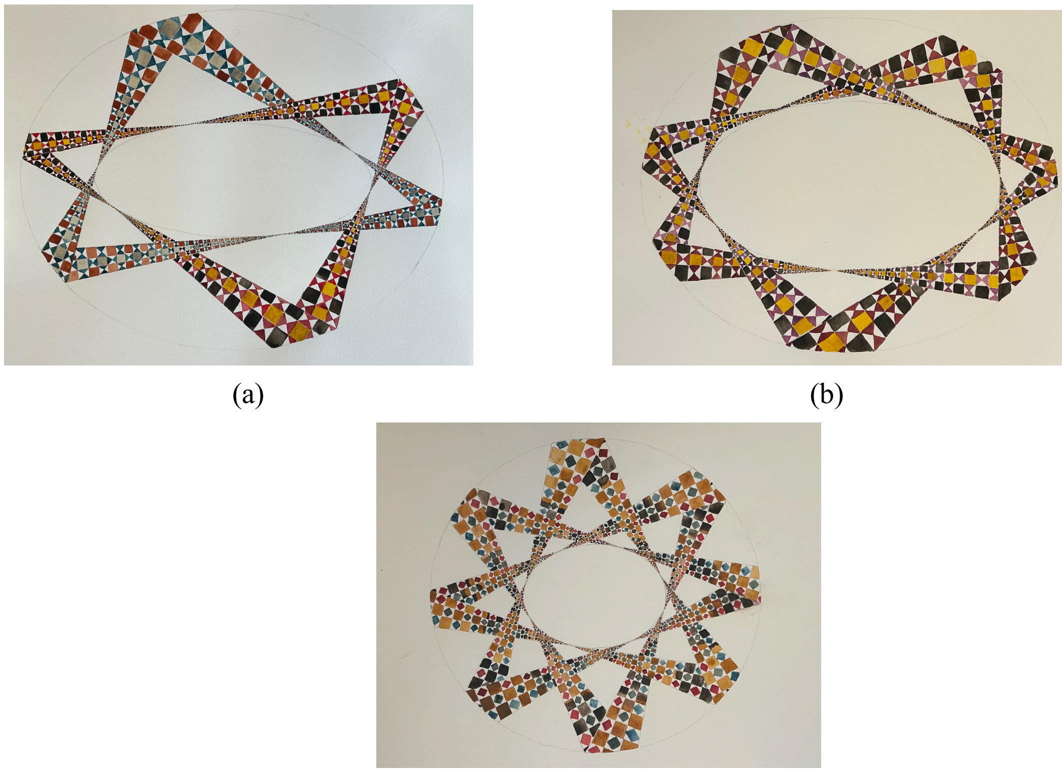
In summery the steps followed are to begin with an ellipse and numerically find a closed path. Then determine the inscribed ellipse. Next is to perturb the initial point and find the new closed paths. These new paths can be found segment-wise either as tangents to the inscribed ellipse or equivalently through the reflection property.



(a) (b) (c)

Figure 3: *(a) Four triangles each tangent to the interior ellipse (b) with orthogonal coordinates added (c) seven-sided polygon.*

The pictures below in Figure 4 provide a sample of such grids with varying numbers of line segments, with ornamentation. Note that in each case the polygons thus constructed are tangent to an inscribed confocal ellipse.

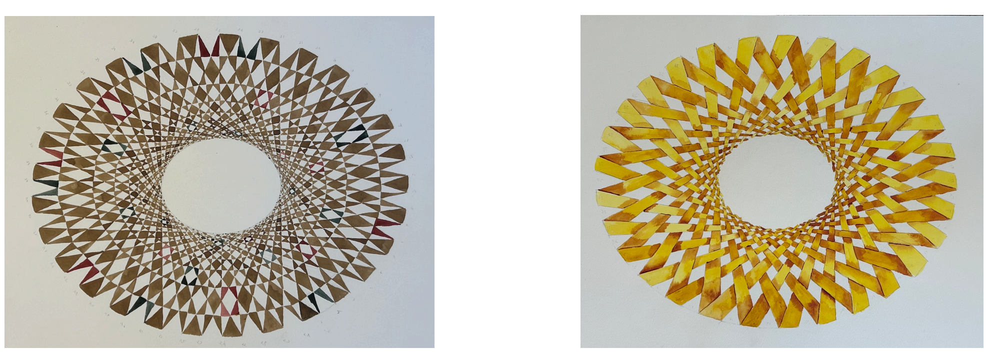


(a) (b)

(c)

Figure 4: Artwork created by starting from an initial (a) pair of triangles, based on Figure 3a, (b) seven-sided polygon, based on Figure 3c and (c) eight-sided polygon.

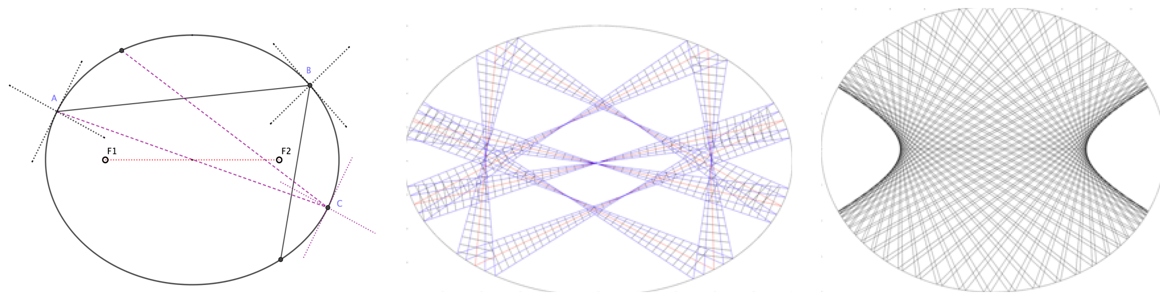
By varying the initial angle, polygons of arbitrary numbers of sides can be generated. Illustrated below in Figure 5 are two examples of polygons made up of 58 segments with patterns based on the self-intersection of the path. The artwork here is derived solely from a single closed path as opposed to a grid.



(a) (b)

Figure 5: (a) tiled pattern (b) weaved pattern. All graphics have been developed in Geogebra® and MATLAB®.

In the examples above the initial angle was chosen so that the line segment did not pass between the foci of the ellipses. If the angle is chosen so that the line passes through the foci, then the resultant caustic will be a hyperbola, as opposed to the ellipse demonstrated above. To clarify, in Figure 6(a) below, the initial solid line AB does not pass through the foci (F1 and F2) and will give rise to an elliptic caustic like those shown above; whereas the dotted line AC passes through the foci and will give rise to a hyperbolic caustic. Figure 6(b) shows the continuation of this hyperbolic case through closure with an associated grid. Figure 6(c) displays a hyperbolic caustic arising from a single orbit. The artwork in Figure 7 is again based on these single paths, as opposed to a grid.



(a) (b) (c)

Figure 6: *(a) demonstration of the beginning of paths producing elliptic (solid) and hyperbolic (dotted) caustics (b) a grid formed from a closed path with hyperbolic caustic and (c) a single closed path with hyperbolic caustic.*

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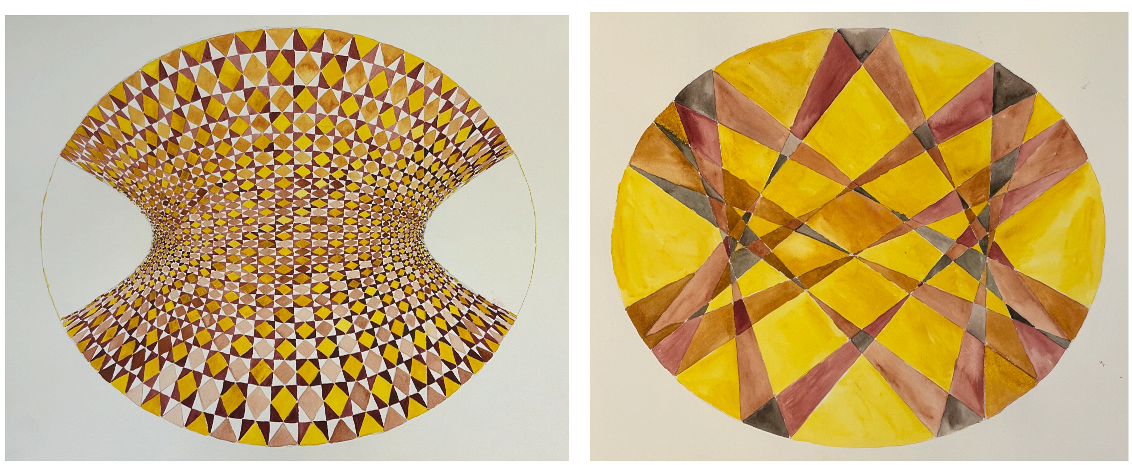

(a) (b)

Figure 7: *Artwork constructed from a closed paths with hyperbolic caustics based on (a) 32 segments and (b) 10 segments.*

References

[1] V. Dragovic and M. Radnovic. *Poncelet Porism and Beyond*. Birkhauser, 2010.

[2] S. Tabachnikov. *Geometry and Billiards*. AMS, 2005