Constructing Triangulations and Polyhedra with Dihedral Symmetry

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Abstract

A planar triangulation is a planar drawing of a maximal planar graph such that any two edges intersect at most at their endpoints and each face is bounded by a cycle of length three contained in the planar graph. In this paper, we investigate the construction of polyhedra arising from maximal planar graphs. In particular, we construct a family of maximal planar graphs with dihedral automorphism groups. Moreover, we demonstrate that these graphs can be realized as polyhedra with congruent triangular faces in the Euclidean 3-space having dihedral symmetry groups. We achieve this result by exploiting Grünbaum-colorings.

Introduction

Over the years, graphs have fascinated both mathematicians and artists. Their combinatorial structures spark curiosity and offer endless possibilities to explore patterns, relationships, and ideas that often inspire creative and visually striking designs, see [5, 6, 12, 20, 22]. A question that often arises in the study of graphs is:

Question. *How can we produce a "nice" drawing of a given graph?*

We refer to a graph drawing as "nice" if it clearly reveals several properties of the corresponding graph through its visual representation. For instance, in this paper, we investigate planar graphs which can be drawn in the Euclidean plane such that any two drawn edges of the given graph are only allowed to intersect at their endpoints. Another desirable property is to have a straight-line planar drawing which Tutte constructs in the case of a given 3-connected planar graph, as discussed in [24]. The study of 3-connected planar graphs turns out to be intriguing, as these are exactly the graphs that are formed by the vertices and edges of a polyhedron, according to Steinitz's theorem [23]. This study becomes particularly interesting when considering maximal planar graphs, i.e. 3-connected planar graphs, where all faces of a planar drawing (called *planar triangulation*) are bounded by cycles of length three known as *3-cycles*. Hence, this class of planar graphs naturally opens up the following question:

Question. *Can a planar triangulation be realized in Euclidean* 3*-space as a polyhedron consisting of congruent triangles as faces?*

This question has been addressed in various works. For instance, Miller (in [14]) resp. Brakhage et al. (in [3]) investigate the planar triangulation that describes the incidence structure of an icosahedron and classify icosahedra with scalene resp. equilateral triangles as faces. Moreover, in [1] and [4] the authors construct polyhedra with prescribed non-trivial symmetry groups. We further refer the reader to [7, 8, 15, 19] for studies on the construction of polyhedra with congruent polygons as faces.

In this paper, we are interested in the construction of highly symmetric polyhedra. In particular, we construct an infinite family of maximal planar graphs and compute corresponding polyhedra whose surfaces consist of congruent triangles and have dihedral symmetry groups. Note that we exploit the computer algebra systems GAP [9] and Maple [13] to utilize the functionalities provided by the packages [18] and [21], respectively. The illustrations in this paper are generated using the GAP package GAPic [17]. In [2] we provide our implementations to construct the maximal planar graphs and the corresponding polyhedra discussed in this paper.

Preliminaries

In this work, we follow the terminology from [16], including concepts such as planar graphs, connectivity and (induced) subgraphs, to name a few. We focus on planar graphs where adding an edge between any two non-adjacent vertices yields a non-planar graph. In the literature, these graphs are called *maximal planar graphs*. It can be observed that such a graph is 3-connected, i.e. the graph remains connected even after removing at most 2 vertices, and can be drawn in the Euclidean plane to obtain a planar triangulation. This planar triangulation subdivides the plane into connected components called faces that are all bounded by 3-cycles. By Whitney (see [26]), it follows that every maximal planar graph has a unique planar triangulation and hence a unique set of triangular faces. Consequently, we define a *triangulation* $T := (V, E, F)$ as a triple, where $G_T := (V, E)$ is a maximal planar graph and F is the set of faces in the corresponding planar drawing of G_T. The *automorphism group* $\mathrm{Aut}(T)$ of a triangulation $T = (V, E, F)$ is the subgroup of the automorphism group of G_T which leaves the set F invariant. Since G_T is 3-connected and planar, we know that $\mathrm{Aut}(G_T) = \mathrm{Aut}(T)$, see [25]. In addition, a triangulation T can be equipped with a Grünbaum-coloring (see [11]). Here, a *Grünbaum-coloring* of $T = (V, E, F)$ is a map $\omega : E \to \{\mathrm{r,g,b}\}$ such that for each 3-cycle forming a face, the corresponding edges are colored differently. Note that a planar triangulation can have more than one Grünbaum coloring.

As an example, we consider the graph $G_T = (V, E)$ that forms a triangulation of the octahedron. In this case, G_T can be drawn as a planar triangulation and can be Grünbaum-colored as illustrated in Figure 1a. By examining the illustration below, we see that the arising faces of G_T are given by $F = \{(v_1, v_2, v_3), (v_1, v_2, v_5), (v_1, v_3, v_4), (v_1, v_4, v_5), (v_2, v_3, v_6), (v_2, v_5, v_6), (v_3, v_4, v_6), (v_4, v_5, v_6)\}$ and $T = (V, E, F)$ forms the corresponding triangulation.

(a) **(b)**

Figure 1: *(a) Triangulation T of the octahedron with a Grünbaum-coloring and (b) polyhedron that corresponds to T.*

We aim to realize certain triangulations as polyhedra with congruent triangles as faces by using corresponding Grünbaum-colorings. To achieve this, we associate each edge color with a chosen edge length and analyze the resulting distance equations. For simplicity, we define Λ as the set that consists of triples $(a, b, c) \in \mathbb{R}^3_{>0}$ satisfying the triangle inequalities: $a + b \geq c$, $a + c \geq b$ and $b + c \geq a$. Hence, let ω be a Grünbaum-coloring and $(\ell_r, \ell_g, \ell_b) \in \Lambda$, where ℓ_r (ℓ_g and ℓ_b) corresponds to the length of an edge that is colored r (g and b, respectively). Then, an (ℓ_r, ℓ_g, ℓ_b)-*embedding* of $T = (V, E, F)$ is a map $\phi : V \to \mathbb{R}^3$

such that all vertices $v_i, v_j \in V$ with $\{v_i, v_j\} \in E$ satisfy the distance equation

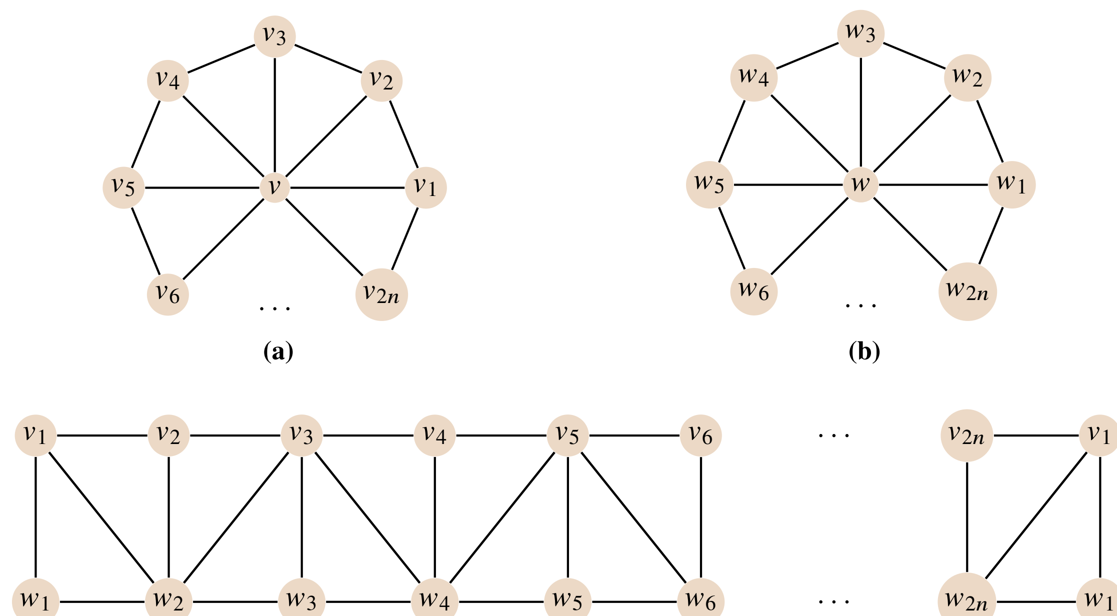
$$\|\phi(v_i) - \phi(v_j)\| = \ell_k,$$

where $\omega(\{v_i, v_j\}) = k$ with $k \in \{r, g, b\}$. By combining the incidence structure of T and the (ℓ_r, ℓ_g, ℓ_b)-embedding ϕ, we obtain a polyhedron in $\mathbb{R}^3$. Here, we call this polyhedron an *embedded triangulation* and denote it by $\mathcal{T}^\phi$. The *symmetry group* of $\mathcal{T}^\phi$ consists of all isometries of $\mathbb{R}^3$ that leave $\mathcal{T}^\phi$ invariant, i.e. $\psi(\phi(V)) = \phi(V)$ for every isometry ψ in the symmetry group of $\mathcal{T}^\phi$. For instance, if T is the triangulation illustrated in Figure 1a, then a $(\sqrt{2}, \sqrt{2}, \sqrt{2})$-embedding ϕ of T is given by $[\phi(v_1), \dots, \phi(v_6)] = [(1, 0, 1), (1, 1, 0), (2, 1, 1), (1, 1, 2), (0, 1, 1), (1, 2, 1)]$. This embedding gives rise to the polyhedron shown in Figure 1b which forms an octahedron in $\mathbb{R}^3$.

Construction of Triangulations with Dihedral Symmetry

In this section, we define an infinite family $(T_{2n})_{n\in\mathbb{N}_{\geq 2}}$ of triangulations such that the automorphism group of T_{2n} is isomorphic to the dihedral group D_{2n} (of order $4n$), for $n \neq 3$. To describe the construction of T_{2n} we exploit wheel and ring graphs illustrated in Figures 2a to 2c. Let T_{2n} be the triangulation containing the vertices $v, w, v_1, \dots, v_{2n}, w_1, \dots, w_{2n}$ such that $G := G_{T_{2n}}$ can be decomposed into the following three induced subgraphs:

1. $G[\{v, v_1, \dots, v_{2n}\}]$ forming a wheel graph, see Figure 2a,
2. $G[\{v_1, \dots, v_{2n}, w_1, \dots, w_{2n}\}]$ forming a ring graph, see Figure 2c and
3. $G[\{w, w_1, \dots, w_{2n}\}]$ forming a wheel graph, see Figure 2b.



(c)

Figure 2: *(a) and (b) wheel graphs with* $2n + 1$ *vertices each and (c) ring graph with* $4n$ *vertices.*

The triangulation T_{2n} is shown in Figure 3 and we observe that for every $n \geq 2$ the triangulation T_{2n} can be equipped with a Grünbaum-coloring. Note that in Figure 2c and in Figure 3 the vertices v_1 and w_1 appear twice. These vertices are drawn twice solely to provide a simplified illustration of the graphs. Hence, these pairs are identified in the combinatorial structure of the corresponding graphs.

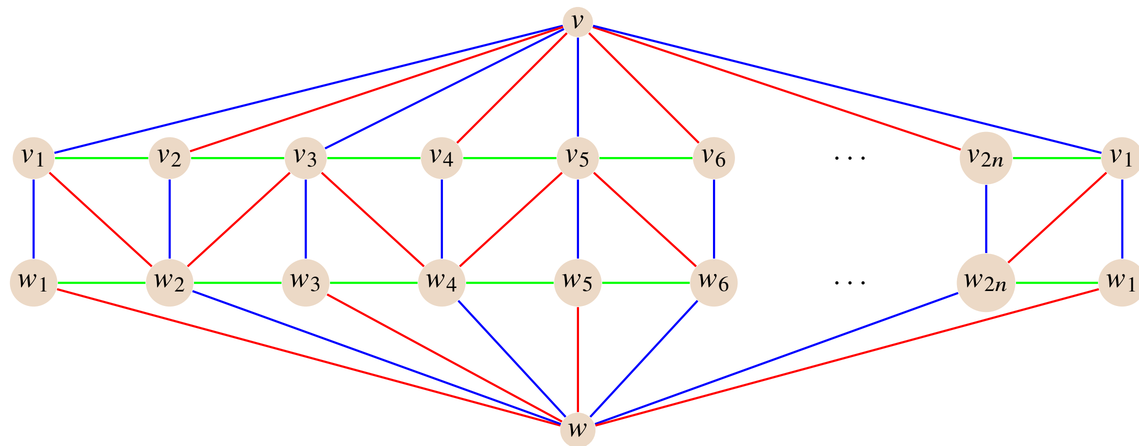


Figure 3: *Triangulation T_{2n} with $4n+2$ vertices and a Grünbaum coloring.*

For $n \geq 2$ and $n \neq 3$, the automorphism group of the triangulation T_{2n} is given by $\mathrm{Aut}(T_{2n}) = \langle \sigma, \tau \rangle$ with the permutations σ and τ defined by

$$\sigma := (v, w)(v_1, w_2, v_3, \dots, v_{2n-1}, w_{2n})(w_1, v_2, w_3 \dots, w_{2n-1}, v_{2n}) \quad \text{and}$$
$$\tau := \prod_{i=1}^{n-1} (v_{1+i}, v_{2n+1-i})(w_{1+i}, w_{2n+1-i}).$$

Since the generators of $\mathrm{Aut}(T_{2n})$ satisfy $\tau^2 = \sigma^{2n} = 1$ and $\tau\sigma\tau = \sigma^{-1}$ and $\mathrm{Aut}(T_{2n})$ contains exactly $4n$ elements, the group $\mathrm{Aut}(T_{2n})$ is indeed a dihedral group of order $4n$. Note that τ is a reflection that fixes v_1, v_{n+1}, w_1 and w_{n+1}. Further, with the help of GAP, we have been able to verify that the triangulation $T_{2\cdot 3}$ has an automorphism group that is isomorphic to $C_2 \times S_4$, which contains a dihedral group of order 12 as a proper subgroup. Moreover, we observe that $T_{2\cdot 3}$ describes the incidences between the vertices and edges of the polyhedron that results from a cube by replacing each of its faces with a square pyramid, see [6].

Construction of Polyhedra with Dihedral Symmetry

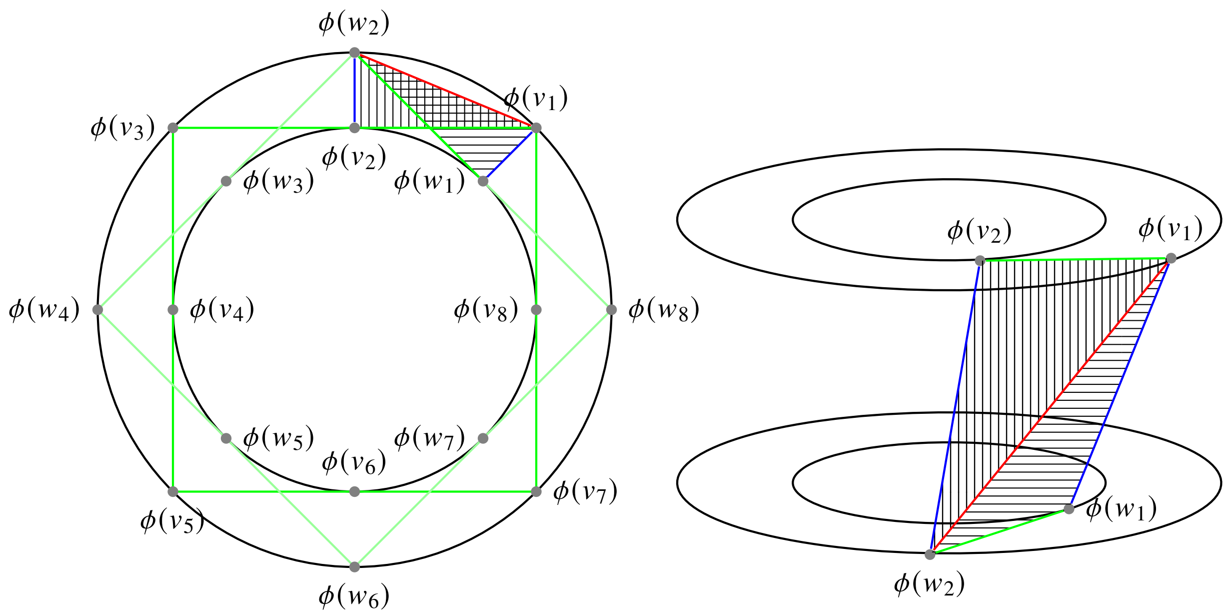
Now, we construct (ℓ_r, ℓ_g, ℓ_b)-embeddings of the triangulation $T_{2n} = (V, E, F)$, where $n \geq 2$ and $n \neq 3$, such that the resulting polyhedra have dihedral symmetry groups of order $4n$. Since the vertices of T_{2n} are given by $v, w, v_1, \dots, v_{2n}, w_1, \dots, w_{2n}$, we can make use of the incidence structure as illustrated in Figure 3.

First, we assign 3D-coordinates to the vertices $v_1, \dots, v_{2n}, w_1, \dots, w_{2n}$ of the ring graph using the following idea: Let $r_1, h_1 \in \mathbb{R}_{>0}$ be positive real numbers, $\alpha := \frac{\pi k}{n}$ for $k \in \mathbb{N}$ with $\gcd(2n, k) = 1$ and $r_2 := |r_1 \cos(\alpha)|$. We aim to place the vertices $v_1, \dots, v_{2n}$ on the xy-plane with $z = h_1$ such that

- the coordinates corresponding to $(v_2, v_4, \dots, v_{2n})$ form an n-gon with corners lying on the circle of radius r_2 centered at $(0, 0, h_1)$ and
- the coordinates corresponding to $(v_1, v_3, \dots, v_{2n-1})$ form an n-gon with corners lying on the circle of radius r_1 centered at $(0, 0, h_1)$.

Similarly, the vertices $w_1, \dots, w_{2n}$ are embedded in the same manner along two concentric circles in the xy-plane, centered at $(0, 0, -h_1)$, by swapping r_1 and r_2. Hence, the coordinates of $v_1, \dots, v_{2n}$ and the coordinates of $w_1, \dots, w_{2n}$ are contained in two parallel planes. We refer to Figure 4 for a visual representation of this construction. The desired embedding of T_{2n} is then completed by assigning 3D-coordinates to the vertices

v, w such that the faces corresponding to the wheel graphs are congruent to the faces corresponding to the ring graph.



(a) **(b)**

Figure 4: *Embedding of the vertices of the ring graph with $n = 4$ and $k = 1$: (a) top view and (b) part of the side view.*

More precisely, for $(\ell_r, \ell_g, \ell_b) \in \Lambda$, we construct an (ℓ_r, ℓ_g, ℓ_b)-embedding $\phi : V \to \mathbb{R}^3$ as follows: The images of the vertices $v_1, \ldots, v_{2n}$ under ϕ can be defined as

$$\phi(v_i) := \begin{cases} (r_1 \cos(i\alpha), r_1 \sin(i\alpha), h_1), & \text{for } i \in \{1, 3, \ldots, 2n-1\} \\ (r_2 \cos(i\alpha), r_2 \sin(i\alpha), h_1), & \text{for } i \in \{2, 4, \ldots, 2n\} \end{cases}$$

and the images of $w_1, \ldots, w_{2n}$ under ϕ as

$$\phi(w_i) := \begin{cases} (r_1 \cos(i\alpha), r_1 \sin(i\alpha), -h_1), & \text{for } i \in \{2, 4, \ldots, 2n\} \\ (r_2 \cos(i\alpha), r_2 \sin(i\alpha), -h_1), & \text{for } i \in \{1, 3, \ldots, 2n-1\}. \end{cases}$$

With $h_2 := \sqrt{4h_1^2 + r_1^2 - 2r_1 r_2}$, we define $\phi(v)$ and $\phi(w)$ as

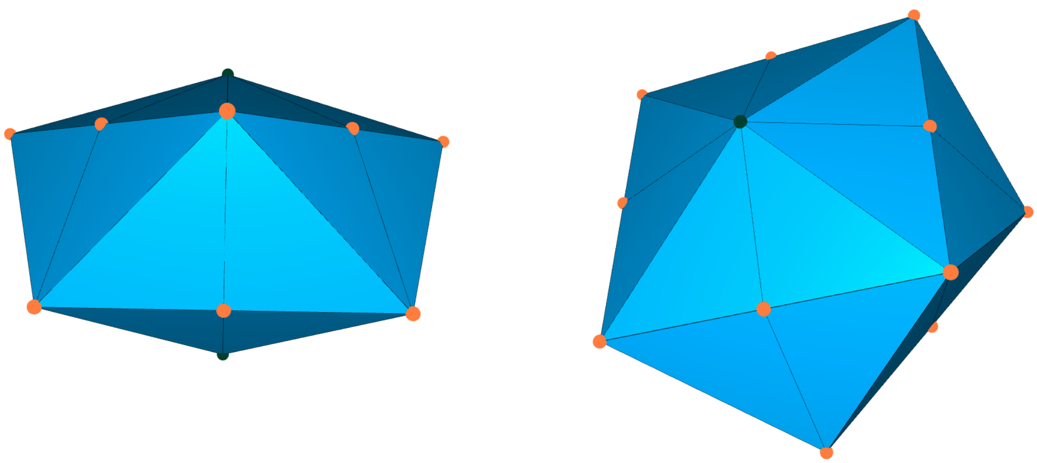
$$\phi(v) := (0, 0, h_1 + h_2) \text{ and } \phi(w) := (0, 0, -h_1 - h_2).$$

By construction, the resulting polyhedron $\mathcal{T}^\phi$ consists of congruent triangular faces and has a symmetry group that is isomorphic to the dihedral group of order $4n$. We observe that the edge lengths of $\mathcal{T}^\phi$ are given by

$$\ell_r = h_2^2 + r_1^2, \quad \ell_g = r_1^2 \sin(\alpha)^2 \text{ and } \ell_b = h_2^2 + r_2^2.$$

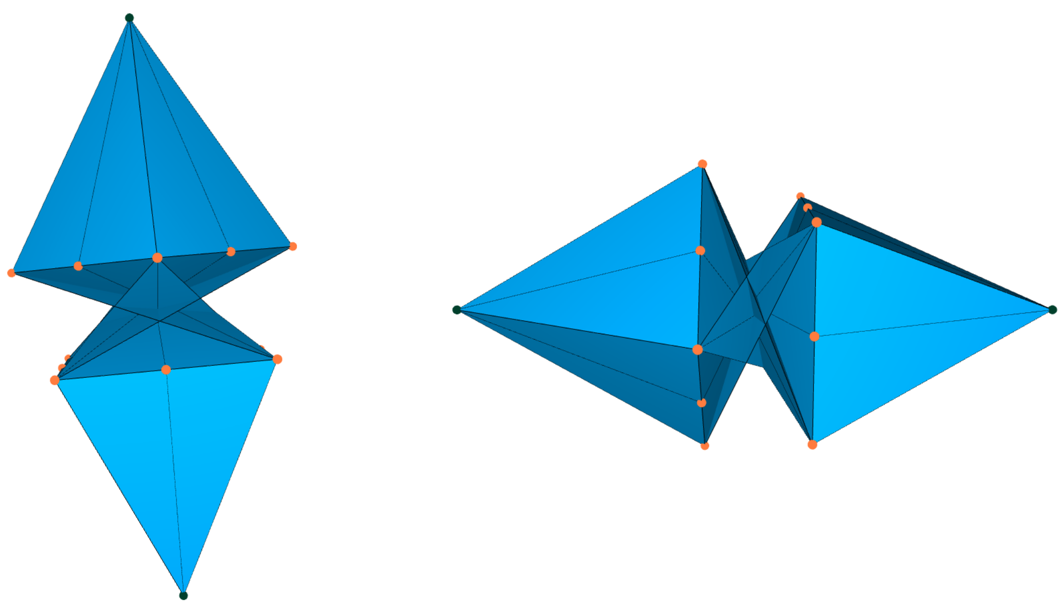
Our construction leads to certain embeddings of T_{2n} where the edge lengths are defined by the parameters k, r_1 and h_1. For given values of r_1 and h_1, our proposed construction yields $\varphi(n)$ distinct (ℓ_r, ℓ_g, ℓ_b)-embeddings of the Grünbaum-colored triangulation T_{2n}, where φ denotes the Euler's totient function. Moreover, the polyhedron obtained from our construction for given r_1, h_1 has self-intersections if $k \notin \{1, 2n-1\}$. Note that the question whether there exists an (ℓ_r, ℓ_g, ℓ_b)-embedding of T_{2n} for every $(\ell_r, \ell_g, \ell_b) \in \Lambda$ remains open.

In Figures 5 and 6, we illustrate different polyhedra that can be constructed by embedding the triangulation $T_{2\cdot 4}$ into $\mathbb{R}^3$ via suitable (ℓ_r, ℓ_g, ℓ_b)-embeddings that are obtained from the above method. In the figures, the vertices v and w are colored in dark green, and the vertices $v_1, \ldots, v_8$ and $w_1, \ldots, w_8$ are colored in orange for better illustration.



(a) (b)

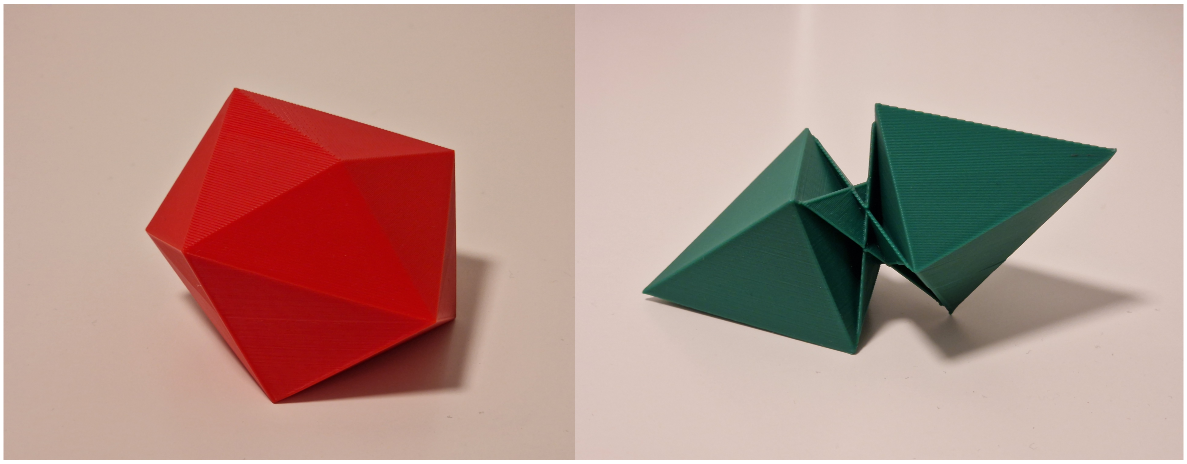
Figure 5: *Different views of the embedding of $T_{2\cdot 4}$ for $k = 1$.*



(a) (b)

Figure 6: *Different views of the embedding of $T_{2\cdot 4}$ for $k = 3$.*

Furthermore, in Figure 7 we present photographs of 3D-printed versions of the above polyhedra. In order to create these models we exploited the GAP package [10].



(a) **(b)**

Figure 7: *3D-printed embeddings of* $T_{2\cdot 4}$ *for* $k = 1$ *and* $k = 3$.

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