Right Angle and Polyhedral String Figures

Karl H. Schaffer

MoveSpeakSpin, Scotts Valley, CA, USA; DanceAndMath@gmail.com

Abstract

String, rope, and thread have been used throughout history for measurement, mechanical applications, fabric creation, and geometrical design. String figures are a kinesthetic and cultural art form practiced with simple loops of string worldwide from ancient times to the present. Although the frequent mathematical claim that ancient Egyptians used knotted rope loops and the Pythagorean triple 3:4:5 to set right angles for construction purposes is not supported by historical evidence, other methods of establishing right angles for architectural use have used rope and string in many ways. We demonstrate how to use geometric string figure techniques to duplicate several of these historical right angle methods. String figures also use simple loops to create beautiful and playful geometric designs. We demonstrate several new ways to use string figure methods with one or more loops to create symmetric polyhedral designs held in many hands.

Introduction

> "There is no indication that the Egyptians had any notion even of the Pythagorean Theorem, despite some unfounded stories about 'harpedenoptai' [rope stretchers], who supposedly constructed right triangles with the aid of a string with 3 + 4 + 5 = 12 knots." – Dirk Jan Struik [19]

As Struik notes all contemporary mathematics historians now agree that once common claims Egyptians used loops with 12 equally spaced knots to lay out a 3:4:5 right triangle to create 90-degree angles in building designs are myth, along with other popular claims about supposed mathematical and astronomical elements hidden in structures like the pyramids (see also [6 Appendix 5]). Although Struik was a committed Marxist persecuted during the McCarthy era he also derogated what he called "primitive" cultures; for a more contemporary description of ancient Egyptian knowledge of the Pythagorean Theorem, see [1, pg. 71]. However, historians also do detail other uses of rope and string in many cultures as aids in architectural design. We will examine several of these and describe ways to use simple loops of string or rope and geometric principles to create right angles, including perhaps provocatively how to use an unmarked loop to easily create a 3:4:5 right triangle! These methods connect closely with ways of using string loops in the ancient art form known as string figures. We will also show several yet unpublished and enjoyable ways to use loops to construct other geometric figures such as string loop polyhedra. Such polyhedral constructions have been popular kinesthetic activities at math circles and other mathematical events for many years.

String Figures, Rope Stretchers, and Right Angles

The art form of string figures consists of imaginative designs created with simple loops of string usually manipulated using the fingers and hands. These designs have been invented and played with by people for thousands of years especially in the most ancient of cultures. Thousands of string figures have been collected, catalogued, and studied since the 19th century, for example see the extensive web sites of the International String Figure Association ISFA [7].

In 1994 the author along with puzzle designer/math educator Scott Kim and dancer Barbara Susco began creating a school outreach performance in the Bay Area, California, titled *Through the Loop: In Search of the Perfect Square* [9]. I had heard that a local string enthusiast, poet, and computer expert Greg Keith had been teaching dancers oversized string figures to be performed on stage, and I invited Greg to join us at a rehearsal. Greg taught us to perform several string figures with large loops of rope, and we also began learning and even creating others, eventually including over 20 in the show that was performed for about ten years. Scott and I also designed several string loop polyhedral forms [11, 12], and Erik Stern and I developed a workshop, "Getting Loopy about Measuring, Graphing, and Geometry" in which participants use large rope loops to kinesthetically explore geometric and scaling properties as they create choregraphed compositions with them [13,14]. Stern and I also created a concert of dances using bungee cords in various configurations [10].

Throughout human history ropes, string, and fiber have been woven, stretched, and manipulated in endless applications [8]. There do exist records of historical uses of string and rope for right angle constructions, some of which we detail in Figure 1. Paulus Gerdes [5, pg. 117] describes how people in parts of Mozambique lay out the base of rectangular homes by stretching two ropes of equal length joined at each of their midpoints, the endpoints of the ropes then becoming the vertices of a rectangle, Figure 1(a). Gerdes also notes [5, pg. 109] that this knowledge is equivalent to the theorem attributed to Thales of Miletus that an angle inscribed in a semicircle is a right angle. Gerdes and Struik both describe the technique used by the Kwakiutl Indians of Vancouver Island in Canada to set the rectangular base of new houses, Figure 1(b) [5, pp 115-116, 19, pg. 48]. A point M is placed at the center of the proposed front of the house and a rope AB is stretched through M along the house front so that MA and MB are equal lengths. Point C is located at the rear side of the house and using other ropes from C to A and from C to B, point C is moved back and forth until lengths CA and CB are equal. This forces CM to be perpendicular to AB. The same technique is used to set other right angles in the construction plan.

Peter Schneider [17] and Richard J. Gillings [6, pg. 208] give detailed descriptions of the ceremonial nature of Egyptian rope stretching methods for constructing square and right-angle architectural plans. Like other contemporary historians they ascribe knowledge of Pythagorean triples to the Babylonians rather than the Egyptians. However, the Egyptians knew accurate approximations for the ratio of the diagonal of the square to its side, for example 17:12, which differs by 0.0025 from its actual value which is the square root of 2. They knew that a figure with dimensions $\frac{17}{12}$ that of a smaller similar figure would have twice its area. This coordinated with the importance in Egyptian mathematics of doubling and halving operations. They apparently used ropes with lengths demarcated perhaps with knots or other means, as seen in the ancient mural in Figure 2 from the Tomb of Menna, to create right isosceles triangle designs as in Figure 1(c) for use in architectural work that required right angles.

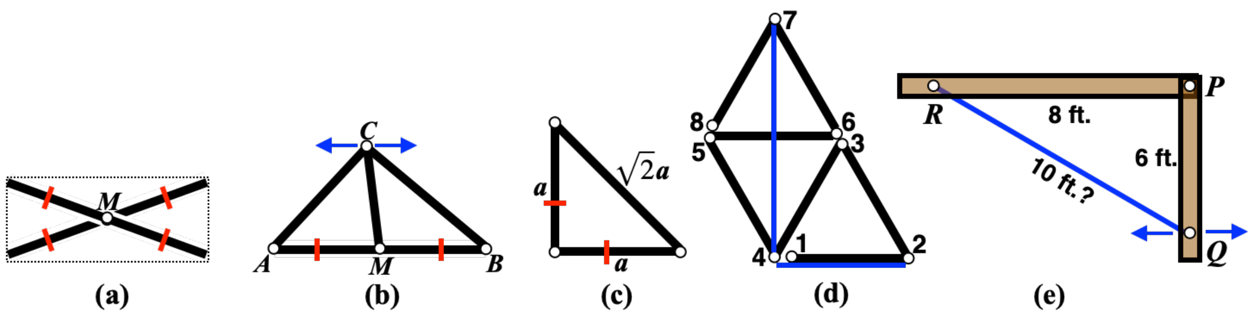

Figure 1: *Historical techniques for construction of right angles. (a) Mozambique. (b) Kwakiutl Indians, Vancouver Island. (c) Egyptian. (d) Mayan. (e) Contemporary construction contractors.*

A Mayan method for obtaining right angles is described in [4] by John Diamontopoulos and Cynthia Huffman. A cord with eight equally spaced knots is stretched out in a plane in sequence 1,2,3,..,8 as shown in Figure 1(d), where we assume that knots 1 and 4, 3 and 6, and 5 and 8 overlap. Then a line shown in blue from knot 7 to 1 will be perpendicular to one from 1 to 2. The Mayans also used the fact that if the diagonals of a rhombus are seen to be equal using an extra piece of rope, then the rhombus is square.

Finally, Richard Salzberg, an actor and sometimes construction contractor, described to me many years ago how contractors commonly use the 3:4:5 triangle to set right angles, Figure 1(e). Two boards shown have a pivot point at P. Along one static board an 8 ft. length PR is measured and along the other, which is allowed to pivot around *P*, a six foot length *PQ* is measured. That board is pivoted left and right until the length *RQ* is found using a tape measure (or perhaps a pre-measured length of rope) to be 10 ft.. This method tends to provide more accuracy than placing a right angle tee at the point *P* and lining up *PQ* along it.

Figure 2: *Rope stretchers, from the Tomb of Menna, Egypt1300 BCE.* Wikimedia Commons [21].

Right Angle String Figures

When Erik Stern and I have taught our "Getting Loopy" or associated workshops for students or teachers we often pose a variety of problems for three or more participants using loops of string or rope. For example, we ask participants to form a quadrilateral that has exactly two parallel sides and seamlessly morph it, without letting it sag, into a hexagon with three pairs of parallel sides – then explain why it is that they have decided the sides must be truly parallel. Or form a convex pentagon and without allowing portions of the rope to become slack, morph it into a five-pointed star (surprisingly tricky!) When asked to find a way to form a right angle with the loop and prove it is such, participants often place the loop next to and line it up with a rectangular tile or other 90-degree architectural feature in the room. Instead, asking that geometric concepts be used to create a provably right angle construction technique makes for a useful workshop activity. We will now describe some string figure techniques for creating right angles that correspond to the methods outlined in Figure 1. Other constructions are also possible.

Figure 3(a) duplicates the Mozambique method in Figure 1(a). It begins with the doubled loop on the left such that when the figure is stretched left to right lengths *MA*, *MB*, *MC*, and *MD* are equal. *A* and C are together stretched outward from *B* and *C*, so that the lengths of *AB* and *CD* are equal. *CD* is rotated upward to the right and stretched so that *BC* and *CD* are straight. The diagonals are now equal with midpoint *M*, which guarantees that *ABDC*'s outline is rectangular, as in Figure 1(a). A similar construction is shown starting on the right of Figure 3(a). Figure 3(b) shows the starting formation for a simpler version using either one or two loops; here the partial loops from the midpoint are stretched next to each other to insure they are equal length.

Figure 3(c) involves three people duplicating the Kwakiutl Indian technique in Figure 1(b). Strand *BD* is rotated up and to the right around pivot point *M*. If points *C* and *D* do not coincide then the third person slides fingers upwards along *MC* and *MD* until *C* and *D* come together, forcing triangle *ABC* to be isosceles with *CM* perpendicular to *AB*. The bottom row of Figure 3(c) shows a similar procedure in case strands *CM* and *CD* overlap when *BD* is opened.

A very simple method to create a right angle is shown in Figure 3(d). Two people stretch a loop between left hands while both right hands grasp adjacent points on the loop as shown. Then extending the two right hand vertices away from each other creates an isosceles triangle with person 2's left hand L_2 holding the midpoint of the side R_1R_2. The two people must now perform a space hold of vertices L_1, L_2, and R_1, which duplicates the process if this loop were to be stretched on the ground with those vertices tacked to the ground. Vertex R_2 is now rotated so that edge L_1R_2 meets vertex L_2 which is then the vertex of a right angle. See the video [16].

Figure 3(e) uses a simple equilateral triangle construction technique we have often seen workshop participants find: three hands expand a loop to a triangle *ABC* and a fourth hand "squeezes" the edges to a

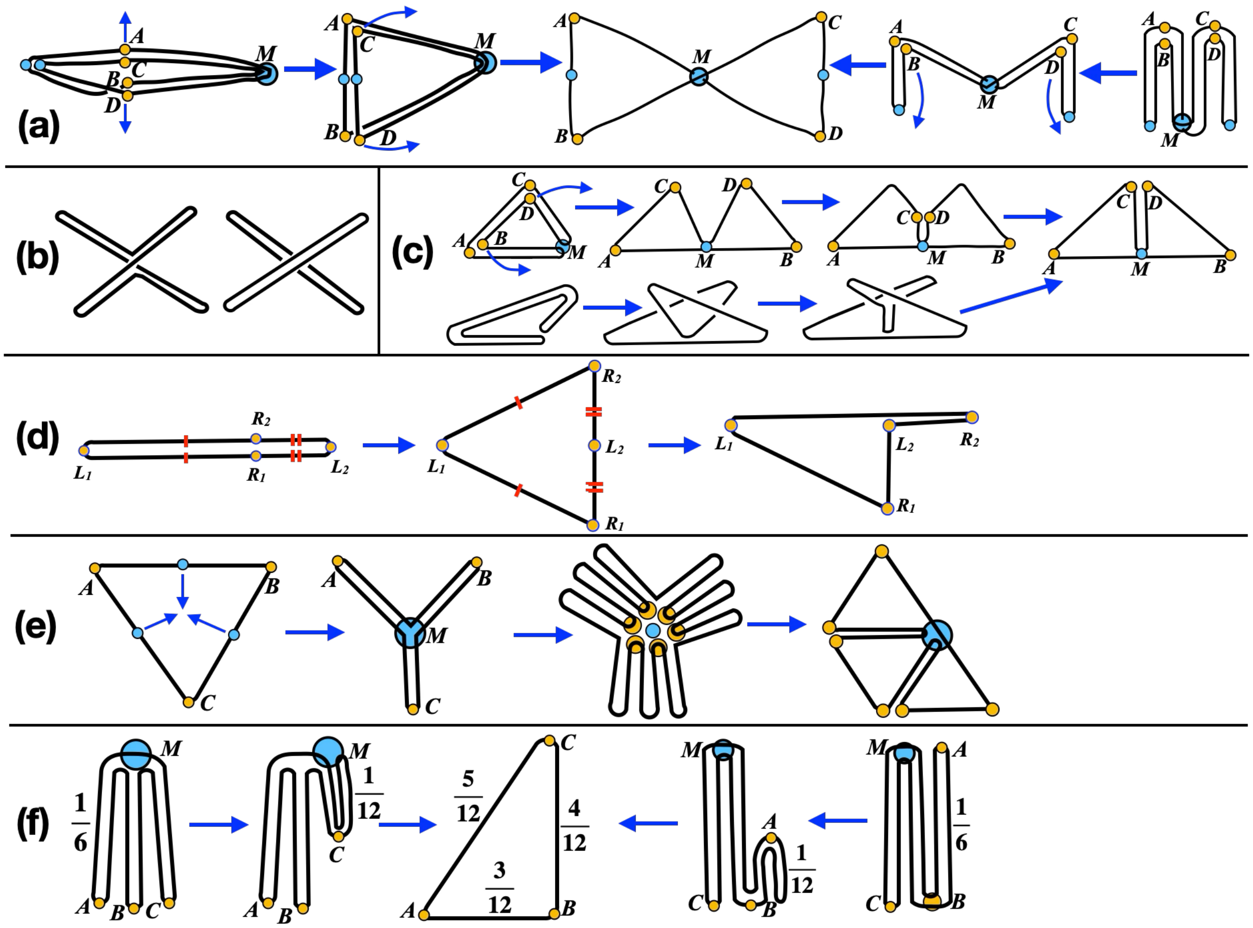

Figure 3: *(a) Mozambique rectangle method. (b) Cross with one or two loops. (c) Kwakiutl right-angle method. (d) Isosceles right triangle or square plus diagonals. (e) Mayan method. (e) 3:4:5 triangle.*

center point *M*. The triangle vertices are easily jiggled to ensure *MA* = *MB* = *MC*. If point *M* is dropped then the expanded triangle *ABC* will be equilateral. However, *MA*, *MB*, and *MC* may be similarly partitioned as shown, each into three partial loops to create three equilateral triangles at point *M*, duplicating the Mayan right-angle design. Of course, three equal loops might be used to create three equilateral triangles instead.

Figure 3(f) details two very similar methods for creating a 3:4:5 right triangle with an unmarked loop. On the left, begin with three equal sized partial loops as described in Figure 3(e). Each vertical strand will now be $\frac{1}{6}$ of the entire loop length. Double back the partial loop *CM* as shown, dividing it in half, so its strands are each $\frac{1}{12}$ of the entire loop. Drop point *M* and expand points *A*, *B*, and *C* to a triangle whose sides will now be $\frac{3}{12}$, $\frac{4}{12}$, and $\frac{5}{12}$ of the entire loop. On the right of Figure 3(f) is shown a slightly different starting point. Interestingly, this demonstrates that any ancient people with string figure expertise and knowledge of the 3:4:5 Pythagorean triple could easily have used an unmarked, unknotted loop to form such a triangle. To repeat, historians now agree that the ancient Egyptians did not use this triangle for architectural designs, yet they did have string figure knowledge [3], as did ancient people all over the world. See [16] for a short video displaying the methods in Figure 3(d)and (f).

Polyhedral String Figures

A variety of methods of using string figure techniques to create polyhedral figures were created by the author and Scott Kim in the 1990s [2, 11 Ch. 9, 12, 18]. Many traditional string figures use one loop, resemble natural images, and are displayed flat, just as traditional origami figures are often created with one sheet of paper, resemble identifiable designs such as animals or people, and are also often displayed flat. However, modular origami often uses multiple identically folded sheets of paper assembled into 3-dimensional forms and displaying many symmetries. I found ways to use multiple string loops to create polyhedra, modeling the idea after modular origami. For example, in [12] I showed how to sequence through the five Platonic solids using six equal sized loops in pairs of three colors and utilizing the solids' symmetries. Other methods, some of which have been passed around "hand to hand" at workshops and math events over the years, are published here for the first time. I will first describe some polyhedral constructions with a single loop. R_i and L_i indicate the right and left hands of person i. Instead of grasping the loop in most cases participants make a ring with thumb and forefinger around the relevant strings, so the strings may slide freely through the ring to adjust the figure. Each vertex is usually represented by one hand. The string loop(s) are usually kept taut rather than allowed to go slack. The mathematical problem here is to decompose the edge-net or "skeleton" of the polyhedron into cycles, with added constraints that the method be not too difficult to accomplish, usually using one hand per vertex, utilize the polyhedron's symmetries, and sometimes allow for sequences from one polyhedron to another.

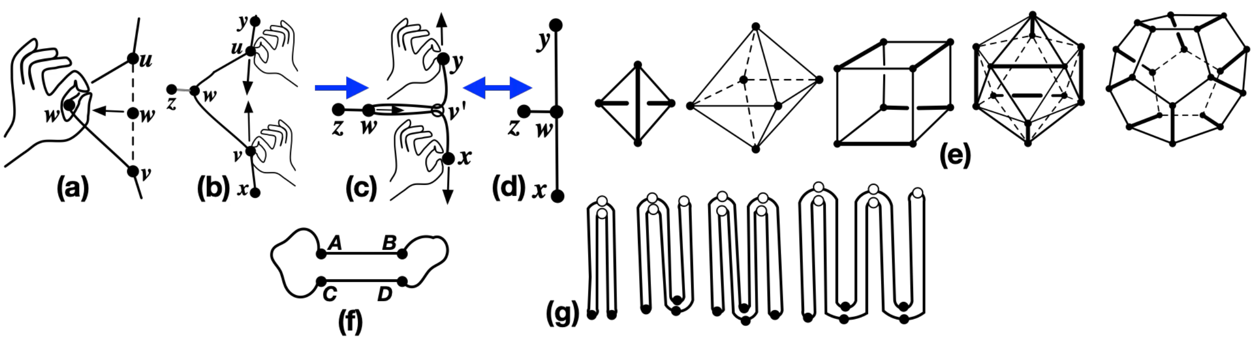

Figure 4: *(a) Edge subdivision. (b) to (c): Vertex amalgamation. (c) to (d): Edge contraction. (d) to (c): Vertex separation and doubled edge creation. (e) Minimal Eulerizations. (f) Matched seqments. (g) Dividing loop into n equal segments for n=2, 3, 4, 5, 6, 8, and 10.*

Figure 4 illustrates graph theoretic descriptions of the processes used as detailed in [12]. Figure 4(a) shows edge subdivision with addition of new vertex w and new edges uw ad vw. Figure 4(b) shows vertex amalgamation resulting in new vertex v' and doubled edge wv'. Figure 4(c) shows edge contraction and vertex amalgamation of w and v'. Figures 4(d) to (c) add doubled edge wv' as two new points on either side of v' are held together while v' is separated with new point w drawn to the left. In Figure 4(e) *minimal Eulerizations* of the Platonic solids with odd degree vertices are shown by thick edges indicating doubled edges.

In addition to graph theoretic processes described above we also assume certain geometric operations are possible:

1. The string loops are inelastic (elastic loops such as bungees are also fun to use for designs! [10])
2. A "space hold" which holds that point's spatial location is possible for any point held on a string loop if slack in the rest of the loop allows while other points are being manipulated.
3. Given point C on string segment AB and enough slack on segment AB we can extend AB through C so that segment AB is a straight line.

4. If enough slack in the loop is available we can create a straight line segment *CD* to match given segment *AB*, see Figure 4(f).
5. We can divide the loop into *n* equal length segments easily for small *n*, see Figure 4(g). From left to right the black dots divide the loops into 2, 3, 4, and 5 equal segments. The black and white dots together divide the same loops into 4, 6, 8, and 10 equal segments.

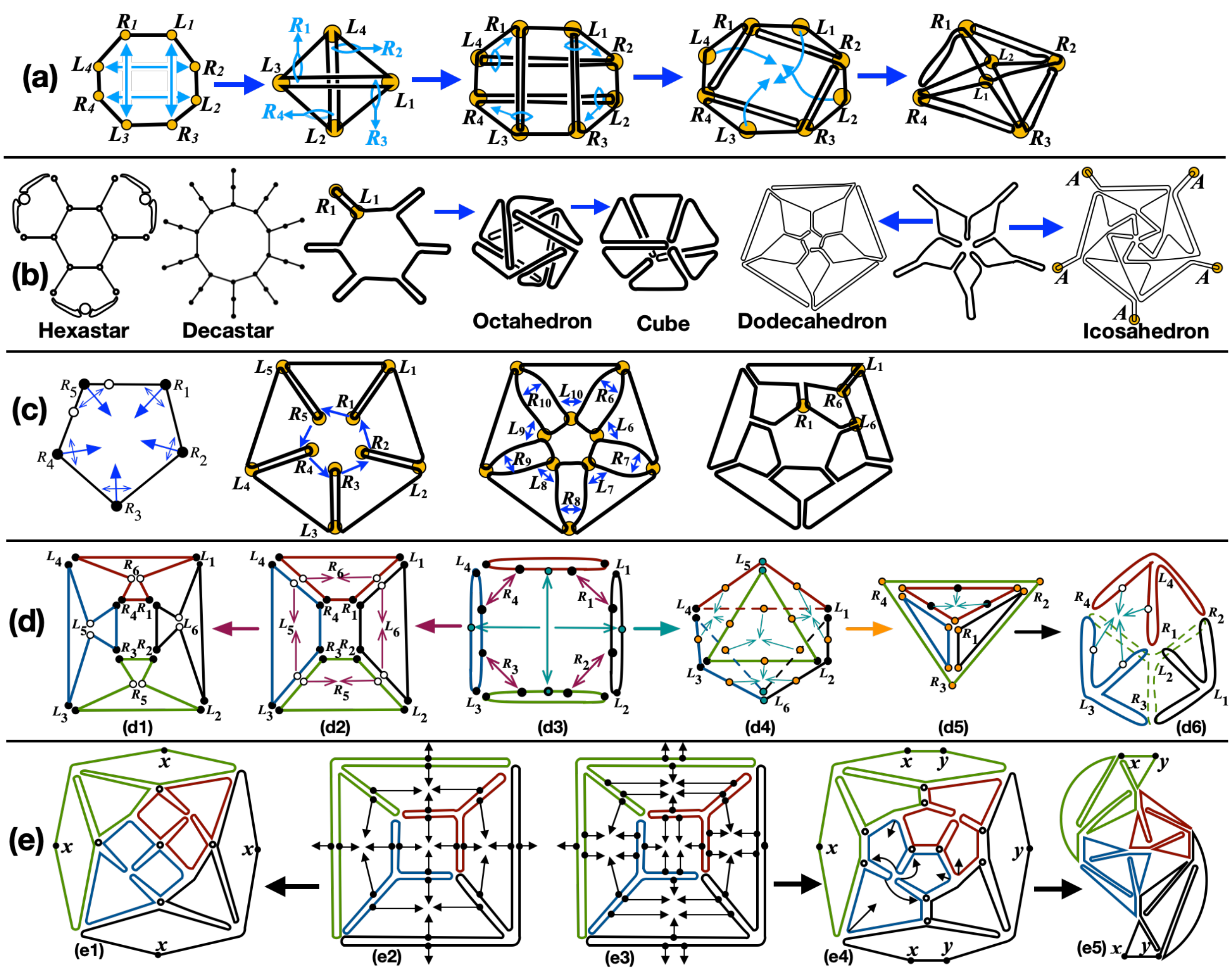

Figure 5: *For larger versions of this Figure see the supplement to this paper in the Bridges Archive. (a) Octagon to tetrahedron to cube to octahedron. (b) Hexastar and decastar octahedron, cube, dodecahedron, and icosahedron. (c) Single loop dodecahedron. (d) 4-loop cube(s), cuboctahedron, octahedron, tetrahedron. (e) 4-loop rhombic dodecahedron, dodecahedron, icosahedron.*

One of the first polyhedral sequences Scott Kim and I came up with was a four-person triangle to tetrahedron to octahedron to cube [11, 2 video of "String Quartet"]. Figure 5(a) shows a four-person octagon to tetrahedron to cube to octahedron sequence, the first part of which is shown in the video [2 "Octaflex"]. Persons 1 and 3 bring their two octagon edges together at the top, persons 2 and 4 at the bottom, and all left hands take the opposite string, forming a tetrahedron. Each right hand now takes the two strands it has handed to the opposite left hand and all hands retreat to their starting octagonal positions, forming a flattened cube. Persons 1 and 3 raise their hands, 2 and 4 lower theirs, to form a cube. Flattening the cube again each left-hand slides left and hands off the doubled partial loop from the opposite right hand to the right hand to its left, and retreats to the left hands' starting positions. L_1 and L_3 carry their vertices above the figure to each other, and L_2 and L_4 carry theirs to each other below, forming an octahedron.

This first sequence can be thought of as using an octagon with four pendant edges, and thus resembles problems posed and solved in [15]. Figure 5(b) shows the “hexastar” used in 2009 in the performance “Harmonious Equations” with Keith Devlin and musical ensemble Zambra, in which all 12 edges were equal length PVC pipe segments used to create the octahedron, cube and two linked tetrahedra on stage. Here we see that as in the previous example the designs can be used to make the cube and octahedron with a single loop manipulated by six people, though fewer may also accomplish them. The “decastar,” a decagon with ten pendant two-edge paths, all edges of the same length, folds into both a regular dodecahedron and a regular icosahedron; ten people can also build these with a large single loop as shown. Participants begin by grasping the loop with the right hand end extending it outward while the left hand circles the base of the point grasped with thumb and forefinger. Five people extend their partial loop upwards, five extend them downwards, shown in the figure second from the right. Each then grasps the doubled string to their right if below or left if above as shown, forming the dodecahedron. The icosahedron begins the same way, but each left hand grabs part of the doubled string to the left if above, right if below, while the right hands extend their partial loops to the center. The vertices labeled *A* are identified, as they come together below the icosahedron.

Figure 5(c) shows a single loop construction of the dodecahedron for ten participants. Five people start with a pentagon held in the right hands. Each right hand extends toward the center as that person’s left hand grasps two points on either side of the right hand. Each right hand grasps one string of the person to the right creating the third diagram in the series. Five more people squeeze two edges together with the right hand and two together with the left as shown. For example person 6 doubles the edge from person 1’s left hand and 6’s left hand doubles the edge from 7’s right hand. This quickly forms the dodecahedron.

Figure 5(d) shows constructions with four loops. Begin at Figure 5(d3) with four loops held in the left hands of four people. When each right hand amalgamates two nearby points on its left hands’ loops a cube is formed, Figure 5(d2). In each of a ring of four squares, as shown by the arrows, two points on opposite edges of the squares are amalgamated in the squares’ centers by the right and left hands of a fifth and sixth person, forming a cuboctahedron, Figure 5(d1). Moving to the right from Figure 5(d3), a fifth person amalgamates two opposite vertices above the loops and two below to form an octahedron, Figure 5(d4). If each of the octahedron’s four triangular loop faces is now “squeezed” by a right hand, meaning the three midpoint vertices from the triangle’s edges are amalgamated in the triangle’s center, the octahedron’s original six vertices held in left hands are released, and the four new vertices are extended outward, a tetrahedron will be formed, Figure 5(d5). Each of the tetrahedron’s four triangular faces is now squeezed by four left hands while the four right hands maintain their hold on the tetrahedron’s four vertices, thus forming a cube, Figure 5(d6).

If each of this cube’s six faces are squeezed and amalgamated to a central vertex without dropping the cube’s eight vertices, as shown by the arrows in face $R_4L_4R_1L_3$, a rhombic dodecahedron will be formed This may be more easily seen from the alternative cube diagram in Figure 5(e2) to Figure 5(e2), where we leave out the hand labels, as there are a number of ways to assign hands and manipulate these figures. Here vertices labeled *x* are identified and brought together “under” the diagram. Figure 5(e3) shows how to amalgamate six vertices in each face of the Figure 5(d6) cube to form the dodecahedron in Figure 5(e4). The arrows within the blue loop in Figure 5(e4) show how each of the loops in Figure 5(e4) can be further manipulated to form the icosahedron. The vertices *x* and *y* are identified here below the diagram.

We also note that in the rhombic dodecahedron diagram there are four vertices with incident strings of one color and four with incident strings of three colors; if these are held and the other vertices released while the strings are stretched out the cube will reappear. The other six vertices shown by the white circles and the black vertex *x* circle are each incident to two strings; if they are held and the others released then an octahedron will appear. Similarly the four 3-color vertices are those of a tetrahedron. Likewise the twelve small white and black circles in the dodecahedron in Figure 5(e4) are those of the icosahedron to its right, and Figure 5(e5) also has four 3-color vertices of a tetrahedron and eight 3- and 1-color vertices of the cube. Each of the four colored loops in the icosahedron encloses four single color triangles, two more triangles shown have edges of three colors, and two triangles with three colors are not shown, but will appear "under" the diagram. The 3-color triangles exhibit all four combinations of the four loop colors. The constructions in Figures 5(c, d, e) were submitted as part of [12] but were edited out due to space constraints so are published here for the first time.

Summary and Conclusions

We have shown two applications of string figures, techniques for constructing right angles without marking the string loops, and the display of various string figure polyhedra, including several transformations of one polyhedra to another. The incorporation of the symmetries of the polyhedra using loops of different colors help make their constructions easier to accomplish and understand. Future work will include creating animations and short videos to accompany the procedures outlined here, since recreating the designs from written instructions, even with accompanying graphics, can be difficult. These designs also make for engaging classroom activities and can lead to entertaining choreographic compositions.

References

[1] R.L. Cooke. *The History of Mathematics: A Brief Course*, 3rd ed. John Wiley & Sons, 2013.

[2] G. Csicsery. Short films on string polyhedra. http://mathdance.org/html/CsicseryShortFilms.html.

[3] W. Cunnington. "Three Egyptian String Tricks." *String Figure Magazine*, June, 1997, pp 20-24, International String Figure Association.

[4] J. C. D. Diamontopolous and C. J. Huffman. "Maya Geometry in the Classroom: Making a Right Angle the Maya Way." Convergance, Aug., 2013. https://old.maa.org/press/periodicals/convergence/maya-geometry-in-the-classroom-making-a-right-angle-the-maya-way

[5] P. Gerdes. *The Awakening of Geometric Thought in Early Culture*. MEP Publications, 2003.

[6] R. J. Gillings. *Mathematics in the Time of the Pharaohs*. Dover, 1982.

[7] International String Figure Association. https://isfa.org/.

[8] F. Jabr. "The Long, Knotty, World-Spanning Story of String." Hakai Magazine, 3/6/2018. https://hakaimagazine.com/features/the-long-knotty-world-spanning-story-of-string/.

[9] K. Schaffer, S. Kim, B. Susco. *Through the Loop: In Search of the Perfect Square*. Dance performance, 1995.

[10] K. Schaffer & E. Stern. *The Bounds of Discovery*. Dance performance, 1999.

[11] K. Schaffer, E. Stern, and S. Kim. *Math Dance with Dr. Schaffer and Mr. Stern: Whole-Body Math and Movement Activities for the K-12 Classroom*. MoveSpeakSpin, 2001.

[12] K. Schaffer. "A Platonic Sextet for Strings." *College Math Journal*, January, 2012, reprinted in *Martin Gardner in the Twenty-First Century*, ed. by Michael Henle and Brian Hopkins, pp 19-24, Math. Assoc. Amer., 2013.

[13] K. Schaffer and E. Stern. "Getting Loopy with Measuring, Graphing, and Geometry." MoveSpeakSpin, 2013, unpublished [contact the author for a copy].

[14] K. Schaffer and E. Stern. "Loopy Dances." *Bridges Conference Proceedings*, Seoul, Korea, Aug. 14–19, 2014, pp. 515-518. http://t.archive.bridgesmathart.org/2014/bridges2014-515.pdf.

[15] K. Schaffer. "Edgy Puzzles." *Gathering for Gardner 13 Exchange Book*, 2018. https://www.gathering4gardner.org/g4g13-exchange-book/.

[16] K. Schaffer. "Two Methods for Making a Right Angle with a Loop." 2025. https://youtu.be/boioszjaVXs.

[17] P. Schneider. "The Puzzle of the First Square in Ancient Egyptian Architecture." *Nexus IV: Architecture and Mathematics*, 2002, pp 207-221. K. Williams and J.F. Rodriguez eds., Kim Williams Books.

[18] I. Stewart. *Cows in the Maze and Other Mathematical Explorations*, Oxford Univ. Press, Oxford, 2010, Ch. 15 & 21.

[19] D. J. Struik. *Concise History of Mathematics*. Dover, 1967.

[20] D. J. Struik. "Stone Age Mathematics." Scientific American, Dec., 1948, pp 44-49.

[21] Wikipedia. https://en.wikipedia.org/wiki/Rope_stretcher.