Linked Penrose Tilings

Ulrich Reitebuch

Institut für Mathematik, Freie Universität Berlin, Germany; Ulrich.Reitebuch@fu-berlin.de

Abstract

Based on N. G. de Bruijn's work on Penrose tilings, we show a novel way to construct five linked Penrose rhombus tilings. If we consider the classical inflation and deflation of Penrose tilings as a parent-child relation, these five tilings can be seen as belonging to five different families being close friends.

The Aperiodic Tilings by Sir R. Penrose

In the 1970s, Sir R. Penrose discovered aperiodic tilings consisting of a small number of prototiles [3]. The first tiling consists of four types of tiles named *pentagon*, *star*, *boat*, and *diamond* (this tiling is also called P1 tiling). The second tiling (P2) has only two types of tiles called *kite* and *dart*. This article makes use of the rhombus tiling known as P3 tiling, consisting of two types of rhombi. The *thin rhombus* has inner angles $\frac{\pi}{5}$ and $\frac{4\pi}{5}$, the *thick rhombus* has inner angles $\frac{2\pi}{5}$ and $\frac{3\pi}{5}$, and all edges in the tiling have the same length.

These three tilings come with inflation and deflation rules, which allow to create closely related coarser and finer tilings. Aperiodicity is achieved by rules, how tiles may be attached to each other—these are often represented by markings like single and double arrows on the edges of the tiles, but it is also possible to create jigsaw puzzle pieces with tabs and blanks enforcing the attachment rules. The three classical Penrose tilings and attachment rules for the rhombus tiling are shown in Figure 1.

Algebraic Theory

First, we recapitulate parts of the algebraic theory of Penrose tilings by N. G. de Bruijn [1]. Throughout this article, we will write $\varphi = \frac{\sqrt{5}-1}{2}$ and $\Phi = \frac{\sqrt{5}+1}{2}$ for the golden ratio. Following the notation of N. G. de Bruijn, the complex fifth unit roots (solving $z^5 = 1, z \in \mathbb{C}$) are written as

$$\zeta = \frac{\varphi}{2} + \frac{\sqrt{3+\varphi}}{2}i,\ \zeta^2 = -\frac{\Phi}{2} + \frac{\sqrt{2-\varphi}}{2}i,\ \zeta^3 = -\frac{\Phi}{2} - \frac{\sqrt{2-\varphi}}{2}i,\ \zeta^4 = \frac{\varphi}{2} - \frac{\sqrt{3+\varphi}}{2}i, \text{ and } \zeta^0 = 1.$$

Where it is convenient, we identify $(x, y) \in \mathbb{R}^2$ with $z \in \mathbb{C}$ in the usual way by $z = x + iy$.

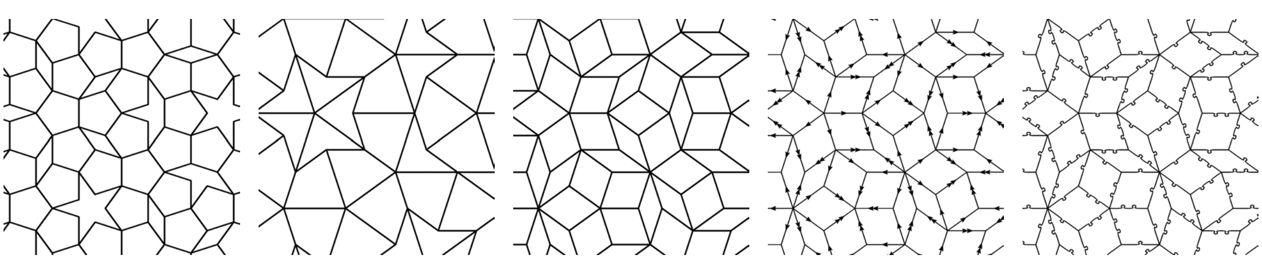

Figure 1: *Penrose tilings — from left to right: P1 tiling, P2 tiling, P3 tiling, P3 tiling with arrows showing attachment rules, P3 tiling as jigsaw puzzle.*

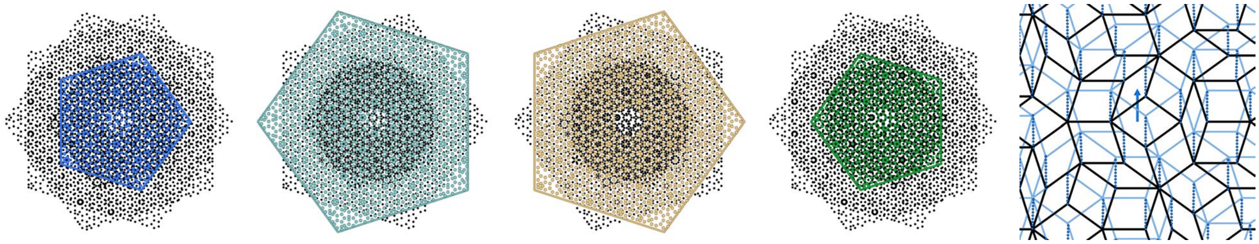

Figure 2: *First four images: Points $\{v\}$ corresponding to the vertices $\{z\}$ of a P3 tiling—from left to right highlighting points in the pentagon area V_1, V_2, V_3, V_4; rightmost image: two shift–equivalent tilings T and $\tilde{T}$, the shift vector is shown as blue arrow.*

To construct a Penrose rhombus tiling, we can use de Bruijn's *cut and project* method: For a given vector $(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4)^\top \in \mathbb{R}^5$, $\gamma_0+\gamma_1+\gamma_2+\gamma_3+\gamma_4 = 0$, we intersect the $2D$ plane X in $5D$ space given by $x^\top(\mu, \nu) = (\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4)+\mu(2, \varphi, -\Phi, -\Phi, \varphi)+\nu(0, \sqrt{3+\varphi}, \sqrt{2-\varphi}, -\sqrt{2-\varphi}, -\sqrt{3+\varphi})$, $\mu, \nu \in \mathbb{R}$, with the standard $5D$ unit cube grid. For each $5D$ cube containing a point x belonging to the plane X in its interiour, we round up each component to the next integer value $k_j = \lceil x_j \rceil$. The point $\hat{x} = (k_0, k_1, k_2, k_3, k_4)^\top \in \mathbb{Z}^5$ is then projected orthogonally to X. These projected points are the vertices of a Penrose rhombus tiling in the plane X, and each Penrose rhombus tiling can be computed in this way.

N. G. de Bruijn shows further, that a Penrose rhombus tiling is fully determined by the complex parameter $\xi = \sum_{j=0}^{4} \gamma_j \zeta^{2j}$. A point $z = \sum_{j=0}^{4} k_j \zeta^j$ with integer representation $(k_0, k_1, k_2, k_3, k_4)^\top \in \mathbb{Z}^5$ is a vertex of the rhombus tiling if and only if its index sum $s = k_0 + k_1 + k_2 + k_3 + k_4$ is not a multiple of 5 and the point $v = \sum_{j=0}^{4} k_j \zeta^{2j} - \xi$ is contained in a pentagon V_r, depending on the residue $r = (s \mod 5)$. Here, V_1 is the regular pentagon with vertices 1, ζ, ζ^2, ζ^3, and ζ^4, V_2 is the regular pentagon with vertices $1+\zeta$, $\zeta+\zeta^2$, $\zeta^2+\zeta^3$, $\zeta^3+\zeta^4$, and ζ^4+1, $V_3 = -V_2$, and $V_4 = -V_1$. Figure 2 shows the points v corresponding to the points z of a rhombus tiling T and the pentagon areas V_r.

Up to now, the plane X is embedded in $5D$ space, and this is not convenient for creating 2D images. By the matrix

$$M = \frac{1}{2}\begin{pmatrix} 2 & \varphi & -\Phi & -\Phi & \varphi \\ 0 & \sqrt{3+\varphi} & \sqrt{2-\varphi} & -\sqrt{2-\varphi} & -\sqrt{3+\varphi} \\ 2 & 2 & 2 & 2 & 2 \\ 2 & -\Phi & \varphi & \varphi & -\Phi \\ 0 & \sqrt{2-\varphi} & -\sqrt{3+\varphi} & \sqrt{3+\varphi} & -\sqrt{2-\varphi} \end{pmatrix}$$

we get a linear transformation $M\hat{x}$, mapping each $5D$ grid point $\hat{x} = (k_0, k_1, k_2, k_3, k_4)^\top \in \mathbb{Z}^5$ to a point $(z_1, z_2, s, p_1, p_2)^\top = M\hat{x} \in \mathbb{R}^5$. The first two entries define a point $z = (z_1, z_2)$ in $2D$, the third entry gives the index sum s, and the last two entries define another point $p = (p_1, p_2)$ in $2D$.

Construction in a Nutshell

Although the cut and project method is described in $5D$ space, it is not necessary to operate in $5D$. To construct a P3 tiling, we first choose the complex parameter ξ. For a section of the $\mathbb{Z}^5$ grid, we use the matrix M to compute for $\hat{x} \in \mathbb{Z}^5$ correspoinding points z and p and the index sum s. If $s \in \{1, 2, 3, 4\}$, we verify if the $2D$ point $v = p - \xi$ is inside the corresponding pentagon area V_s, and if it is, the point z is a vertex of the P3 tiling, otherwise it is discarded. Pairs of points with distance 1 are connected by an edge.

Wieringa Roof

The first three entries (z_1, z_2, s) give a representation by rhombi in $3D$ space—if we scale the third entry by $\frac{1}{2}$, all these rhombi with vertices $\left(z_1, z_2, \frac{s}{2}\right)$ in $3D$ space have the same shape with ratio Φ between their long

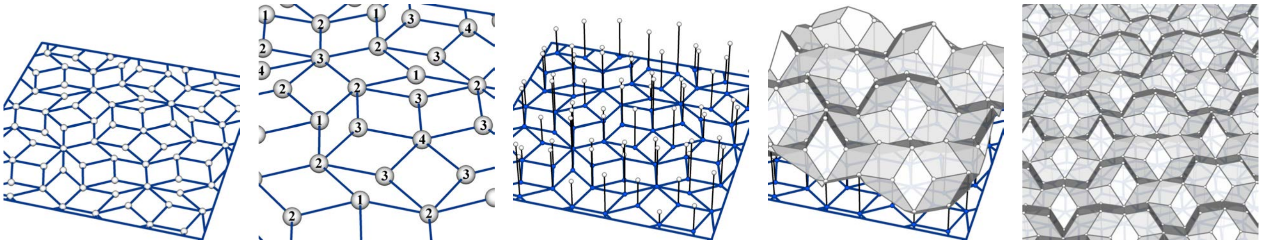

Figure 3: *From left to right: Small part of a P3 tiling, labeling of vertices by index sums, using labels to compute height values $\frac{s}{2}$, Wieringa roof over the tiling, larger part of the Wieringa roof.*

and short diagonal, building a structure called "Wieringa roof" by N. G. de Bruijn [1]. The rhombus tiling with vertex labels and the Wieringa roof are shown in Figure 3. N. G. de Bruijn states about the index vector $(k_0, k_1, k_2, k_3, k_4)$ for the projected $5D$ grid points: "Needless to say since $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$, the sum $k_0 + k_1 + k_2 + k_3 + k_4$ can always be reduced modulo 5, but the fact that the sum is never a multiple of 5 is remarkable." Several grid points $\hat{x} = (k_0, k_1, k_2, k_3, k_4)^\top$ may be projected to the same position (z_1, z_2) in $2D$, but for all grid points projected to the same $2D$ position, the index sum will have the same residue modulo 5. It is not necessary to test all $5D$ grid points, testing grid points with index sum $s \in \{1, 2, 3, 4\}$ is sufficient.

Shift–Equivalence

Two rhombus tilings T for ξ and $\tilde{T}$ for $\tilde{\xi}$ are *shift–equivalent* if and only if $\xi - \tilde{\xi} \in P$, here P is the ideal of all $n_0 + n_1\zeta + n_2\zeta^2 + n_3\zeta^3 + n_4\zeta^4$, with $(n_0, n_1, n_2, n_3, n_4)^\top \in \mathbb{Z}^5$ and $n_0 + n_1 + n_2 + n_3 + n_4 = 0$. Shift–equivalent tilings can be transformed into each other by a global translation [1].

For example, if we compute two tilings T for ξ and $\tilde{T}$ for $\tilde{\xi} = \xi + 5$, since $5 = 4\zeta^0 - \zeta^1 - \zeta^2 - \zeta^3 - \zeta^4 \in P$, we get two shift–equivalent tilings: translating the tiling $\tilde{T}$ by -5 in ζ^0 direction (i.e., along the real axis), we get the same tiling as by computing T for ξ. Since each vertex $\tilde{z}$ of the tiling $\tilde{T}$ has a representation $\tilde{z} = \sum_{j=0}^{4} \tilde{k}_j \zeta^j$, its corresponding vertex z in T can be computed as $z = (\tilde{k}_0 - 5)\zeta^0 + \sum_{j=1}^{4} \tilde{k}_j \zeta^j$, which is a different representation for the same point $z = \sum_{j=0}^{4} k_j \zeta^j$ computed for the tiling T. For the index sum $\tilde{s} = \tilde{k}_0 + \tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3 + \tilde{k}_4$ in the tiling $\tilde{T}$, we have $\tilde{s} \in \{1, 2, 3, 4\}$, in the representation computed from shifting the tiling $\tilde{T}$, we get an index sum in $\{-4, -3, -2, -1\}$, which still—as de Bruijn remarks—omits index sums that are multiples of 5. The rightmost image in Figure 2 shows two tilings T and $\tilde{T}$ with $\xi - \tilde{\xi} = \zeta + \zeta^2 - \zeta^3 - \zeta^4$. The resulting tilings are shift–equivalent; shifting $\tilde{T}$ by $(\varphi\sqrt{2 - \varphi})i$ gives the tiling T.

Relations between not Shift–Equivalent Tilings

In this section, we observe new relations between some not shift–equivalent tilings for parameters ξ and $\tilde{\xi}$, where $\xi - \tilde{\xi} = \sum_{j=0}^{4} n_j \zeta^j$, $(n_0, n_1, n_2, n_3, n_4)^\top \in \mathbb{N}^5$, $\xi - \tilde{\xi} \notin P$. For convenience, we will later choose $\tilde{\xi} = \xi + n$, $n \in \mathbb{Z}$, but the same observations are true for a shift by any integer combination $\sum_{j=0}^{4} n_j \zeta^j$. Using a real integer difference between ξ and $\tilde{\xi}$ is convenient, because in that case the necessary translation direction and distance coincides with $\xi - \tilde{\xi}$.

In the previously described construction, to compute a tiling from ξ, we only use grid points $\hat{x} \in \mathbb{Z}^5$ with index sums $s \in \{1, 2, 3, 4\}$—let us call the resulting Tiling T_0. But in fact, for the $\mathbb{Z}^5$ grid, the points where s is a multiple of 5 are not special. All grid points in the $\mathbb{Z}^5$ grid look exactly the same. We might as well look for grid points $\hat{x}$ with index sum $s \in \{2, 3, 4, 5\}$ and verify for the corresponding points v if they are contained in the pentagon area $V_{(s-1)}$ to construct a new P3 tiling—let us call this Tiling T_1. If we use the $3D$ representation using vertices $(z_1, z_2, \frac{s}{2})$, this new rhombus tiling T_1 will share some vertices with the

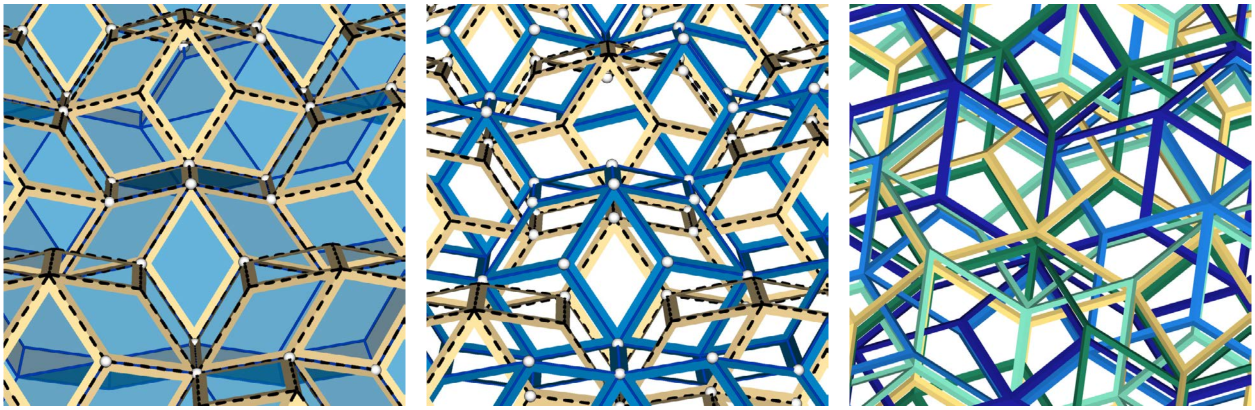

Figure 4: *From left to right: Tilings T_0 (blue) and T_1 (sand) with heights $\frac{s}{2}$ and common points highlighted, same tilings with T_1 translated by $-\frac{1}{2}$ in z–direction, Wieringa roofs for all five linked rhombus tilings.*

$3D$ representation of the original rhombus tiling T_0. Since T_0 does not contain vertices with index sum $s = 5$ and T_1 does not contain vertices with index sum $s = 1$, they only share points with index sums $s \in \{2, 3, 4\}$. The tilings T_0 and T_1 share even some edges and complete rhombi. If we translate the $3D$ representation of T_1 down to the usual range for Wieringa roofs $\left[\frac{1}{2}, \frac{4}{2}\right]$, using vertices $\left(z_1, z_2, \frac{s-1}{2}\right)$, the tilings T_0 and T_1 still have vertices with shared $2D$ position (z_1, z_2), but now the height values are always different for the two tilings. So in the $3D$ representation, the two tilings do not have shared vertex positions, and even edges of the two tilings do not intersect in $3D$. There are however edges intersecting rhombi of the other tiling, and intersections of rhombi of the two tilings. The centre parts of the rhombi can be cut out, such that the two tilings with missing interiour parts of all rhombi do not intersect in $3D$. Figure 4 shows tilings T_0 and T_1 with both variants of height values. In the same way, we can now create more tilings T_n. First we look for grid points with index sums $s \in \{1 + n, 2 + n, 3 + n, 4 + n\}$, verify for each point $\hat{x}$ if its point $v = p - \xi$ is contained in the pentagon area V_{s-n}, and in that case compute the corresponding point $(z_1, z_2, \frac{s-n}{2})$ for the tiling T_n. The tiling T_5 is identical to T_0, and in general $T_n = T_{n+5}$. So this way, we get five different tilings, each of them omitting one residue modulo 5 in the index sums of its vertices.

If we compute a tiling T_n for $\tilde{\xi} = \xi + n$ and then translate the tiling by $-n$ along the real axis, this results in the same tiling as looking for points $v = \sum_{j=0}^{4} k_j \zeta^{2j} - \xi$ with index sum $s = r - n$ in the pentagon area V_r and computing the corresponding tiling points $z = \sum_{j=0}^{4} k_j \zeta^j$. Since $n \notin P$ unless n is a multiple of 5, the five tilings T_0, T_1, T_2, T_3, and T_4 are not shift–equivalent to each other, but they are very closely related—let us call them *friends*. Starting with the tiling for one of the $\xi + n$ values and computing its friend tilings results in exactly the same set of five friend tilings—they are just all five translated by n. So within these five friend tilings, every one can be used as the start to generate the same set of five friend tilings.

Friendship between Families

Each P3 tiling is part of a family of P3 tilings by the *substitution rule*, which is shown in Figure 5: For the rhombus tiling, we get a finer rhombus tiling by splitting each thin rhombus into two halves of small thick rhombi and two halves of small thin rhombi, and splitting each thick rhombus into two halves of small thin rhombi, two halves of small thick rhombi, and one complete small thick rhombus. The new half thick and thin small rhombi always have a fitting other half small rhombus in the neighbouring coarse rhombus, because all coarse edges with single arrows look the same and all coarse edges with double arrows look the same in the substitution rule. In reverse, it is also possible to get back to a coarser tiling from a fine tiling: whenever

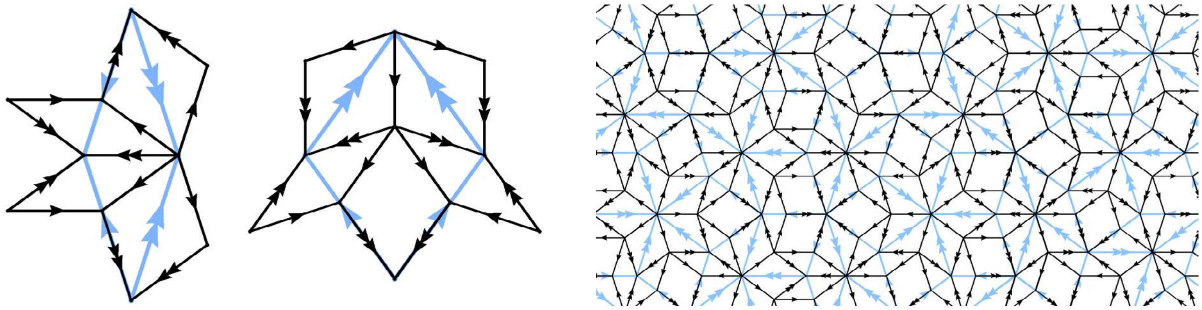

Figure 5: *The refinement rules for the P3 rhombi and a section of a parent tiling (light blue) and its child tiling (black), both with arrows indicating refinement rules on their edges.*

two fine thin rhombi share an edge, this is a double arrowed edge, and it is the short diagonal of a coarse thin rhombus, and every single arrow between two thick rhombi in the fine tiling is part of the long diagonal of a coarse thick rhombus.

The ratio of fine to coarse edge length is φ. By de Bruijn's method, it is also possible to compute finer and coarser tilings directly: For a given tiling T computed from the complex parameter ξ, we can compute the next finer tiling (its "child" by the substitution) by computing a tiling from the complex parameter $\xi_c = -\varphi\xi$ and scaling the resulting tiling by $-\varphi$. The coarser "parent" tiling which has T as its child can be created by computing a tiling for the complex parameter $\xi_p = -\Phi\xi$ and scaling it by $-\Phi$. For a given tiling T_0, we can now consider its parent tiling P_0 and child tiling C_0 and compute their friend tilings P_1, P_2, P_3, P_4, C_1, C_2, C_3, and C_4 in the same way as for T_0, by using the same ξ_p and ξ_c, verifying the v points with the pentagon area V_r for index sum $s = r - n$ and using the same scaling by $-\varphi$ and $-\Phi$ for the resulting friend tilings of the child and parent tiling. The friend tilings of child tiling C_0 turn out to be the child tilings of the friend tilings of T_0, so friendship is kept within families: The tiling C_1 is the child of friend tiling T_2, C_2 is the child of friend tiling T_4, C_3 is the child of friend tiling T_1 and C_4 is the child of friend tiling T_3. The same permutation links the friends of T_0 to the friends of its parent tiling P_0: The tiling P_2 is the parent of friend tiling T_1, P_4 is the parent of friend tiling T_2, P_1 is the parent of friend tiling T_3 and P_3 is the parent of friend tiling T_4. This permutation occurs because the friend tilings on the parent level and on the child level relative to T are scaled by $-\Phi$ and φ, and the friends can be computed by integer shifts in ξ—but since $\zeta + \zeta^4 = \varphi$ and $\zeta^2 + \zeta^3 = -\Phi$, this stays still in the set of friends and only permutes the ordering of friends in different generations.

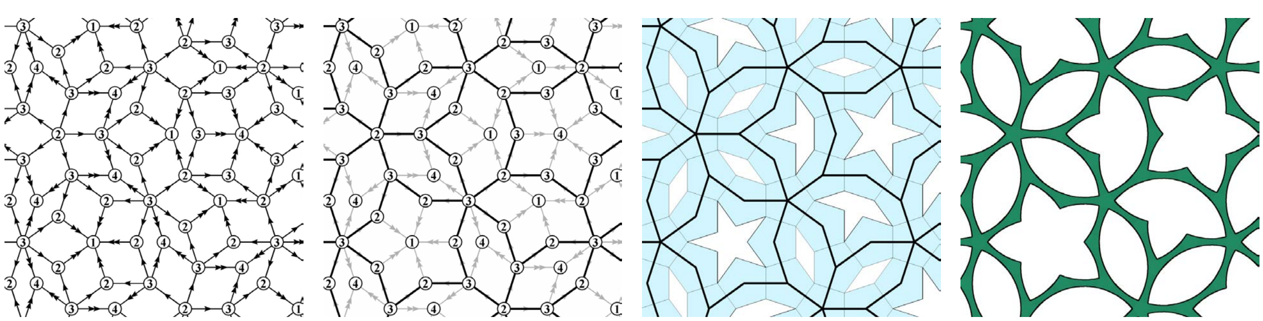

Figure 6: *The starfish / ivy leaf / hexagon tiling. From left to right: P3 tiling with labeled vertices, rhombus tiling with labels* 1 *and* 4 *removed, starfish / ivy leaf / hexagon tiling in relation to the P1 tiling, used representation of the starfish / ivy leaf / hexagon tiling with circular cut lines.*

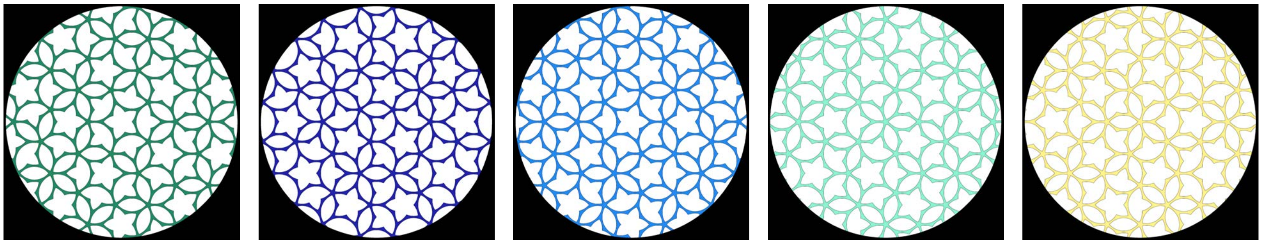

Figure 7: *Five friend starfish / ivy leaf / hexagon tilings, which will be interwoven with each other.*

Artistic Realisation

It is hard to see the five tilings in the complete rhombus tilings with cut–out rhombus centres. Therefore, for a realisation of five linked Penrose tilings made from paper, we use a reduced tiling, named starfish / ivy leaf / hexagon tiling by E. A. Lord [2]. We get this tiling by removing all vertices with index sums 1 and 4 from the rhombus tiling, keeping only edges between vertices with index sums 2 and 3. This way, groups of five thick rhombi (related to a P1 star) are joined into a *starfish*, three thick rhombi join with one thin rhombus (related to a P1 boat) are joined into an *ivy leaf*, and two neighbouring thin rhombi (related to a P1 diamond) are joined with one thick rhombus into a *hexagon*. The edges of the starfish / ivy leaf / hexagon tiling can also be obtained by connecting the centres of neighbouring pentagons in the corresponding P1 tiling. Figure 6 shows the starfish / ivy leaf / hexagon tiling in relation to the P1 and P3 tiling.

We cut holes into each starfish, ivy leaf, and hexagon, to let the edges of the related other four tilings pass through the interiours of the tiles. For a visually pleasing appearance, these holes are cut along circular arcs. Figure 7 shows sections of five related tilings. The information about which tiling should be on top and bottom at the vertices is induced from the Wieringa roof s values—since all vertices with values 1 and 4 were removed to obtain the starfish / ivy leaf / hexagon tiling, there are only values 2 and 3 left, so not more than two tilings have a vertex at the same $2D$ position (z_1, z_2). For each pair of tilings with neighbouring shifts n and $n+1$, where they have common vertices, the heights are always different, and in each pair always the same tiling has the top vertex. Figure 8 shows in the top row the five neigbouring pairs of starfish / ivy leaf / hexagon

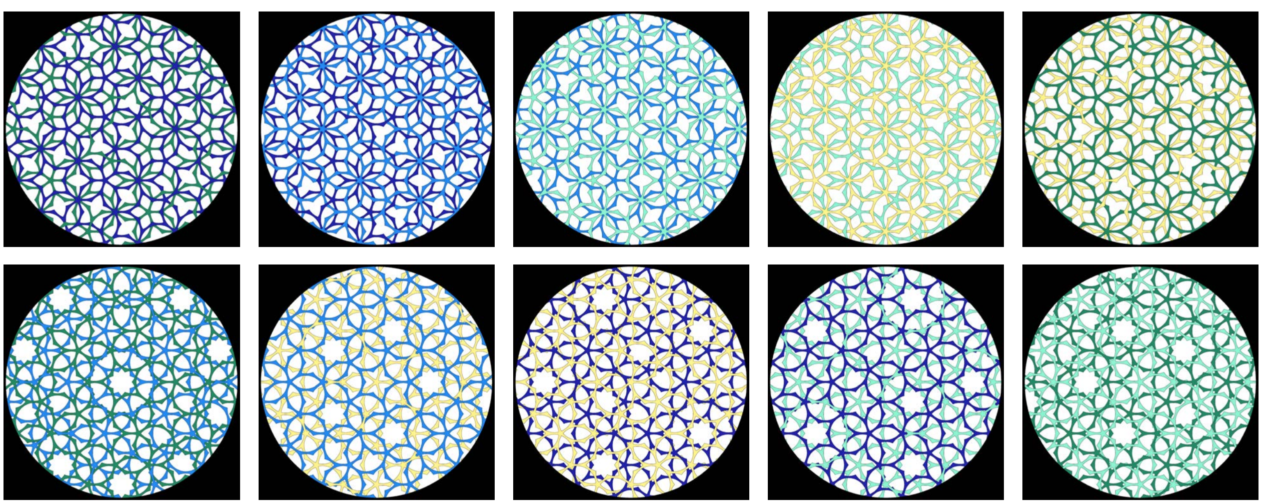

Figure 8: *Top row: The five neighbouring pairs of starfish / ivy leaf / hexagon tilings—for each pair, at common vertices always the same tiling is on top. Bottom row: The five not neighbouring pairs of starfish / ivy leaf / hexagon tilings—there are no common vertices.*

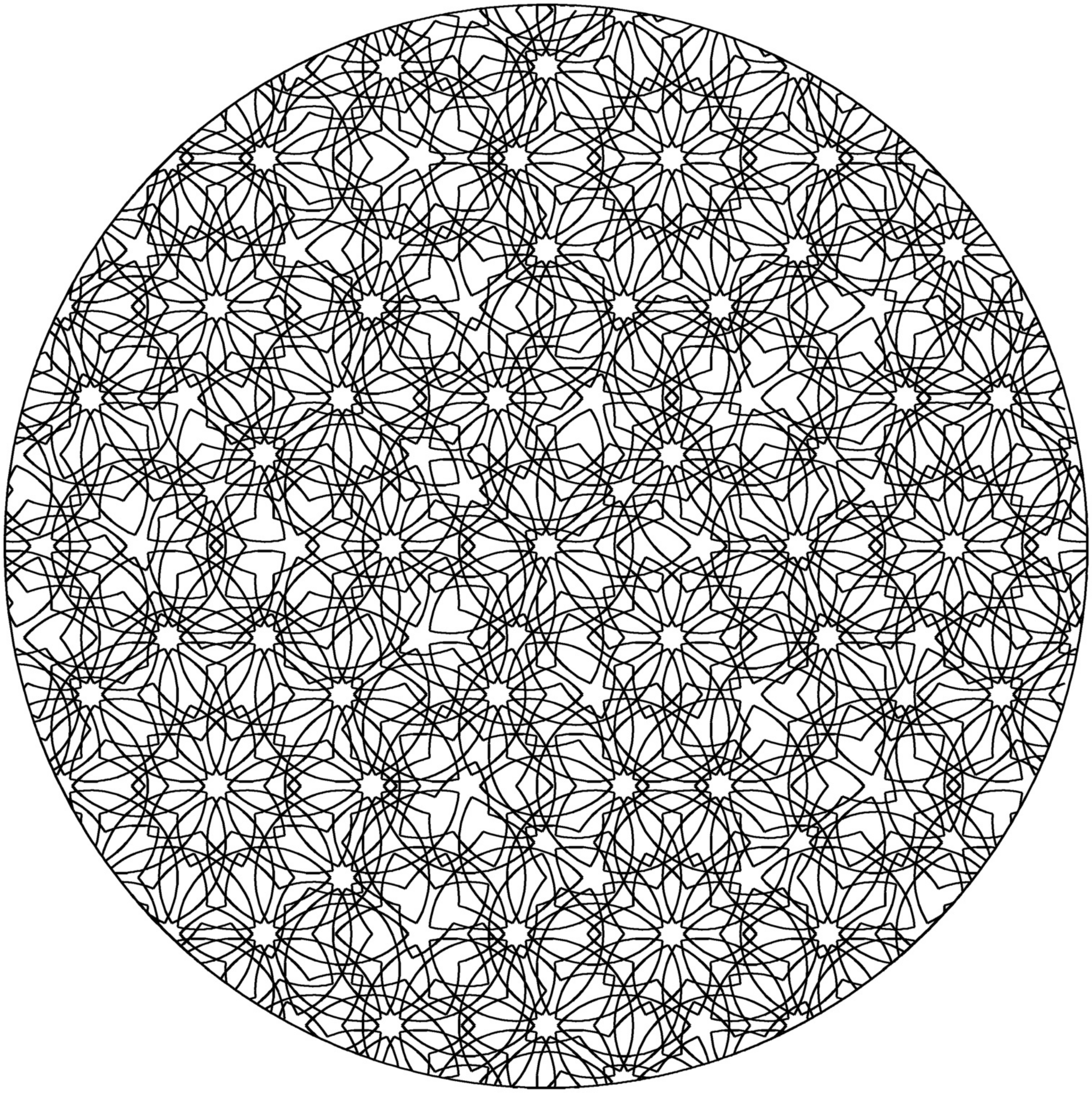

Figure 9: *All cut lines for all five friend tilings.*

tilings. Some of the vertices of the tiling on top, which are not contained in the neighbouring tiling, are below edges of the bottom tiling and vice versa, so the edge graphs of neighbouring tilings are linked into each other. For each of the five pairs of starfish / ivy leaf / hexagon tilings with not neighbouring shifts, there are no common vertices; these five pairs are shown in the bottom row in Figure 8. Also here, the edge graphs of all pairs of tilings are linked into each other. At all edge crossings for all pairs of tilings, the information about which edge is on top is induced by linear interpolation of vertex heights along the edges.

These five sections were realised from five sheets of paper in five different colours and then linked into each other by cutting and gluing—all cut lines for holes in starfish, ivy leaf, and hexagon tiles for all five friend tilings are shown in Figure 9, the final result is part of the Bridges 2025 Exhibition of Mathematical Art, Craft, and Design and is shown in Figure 10.

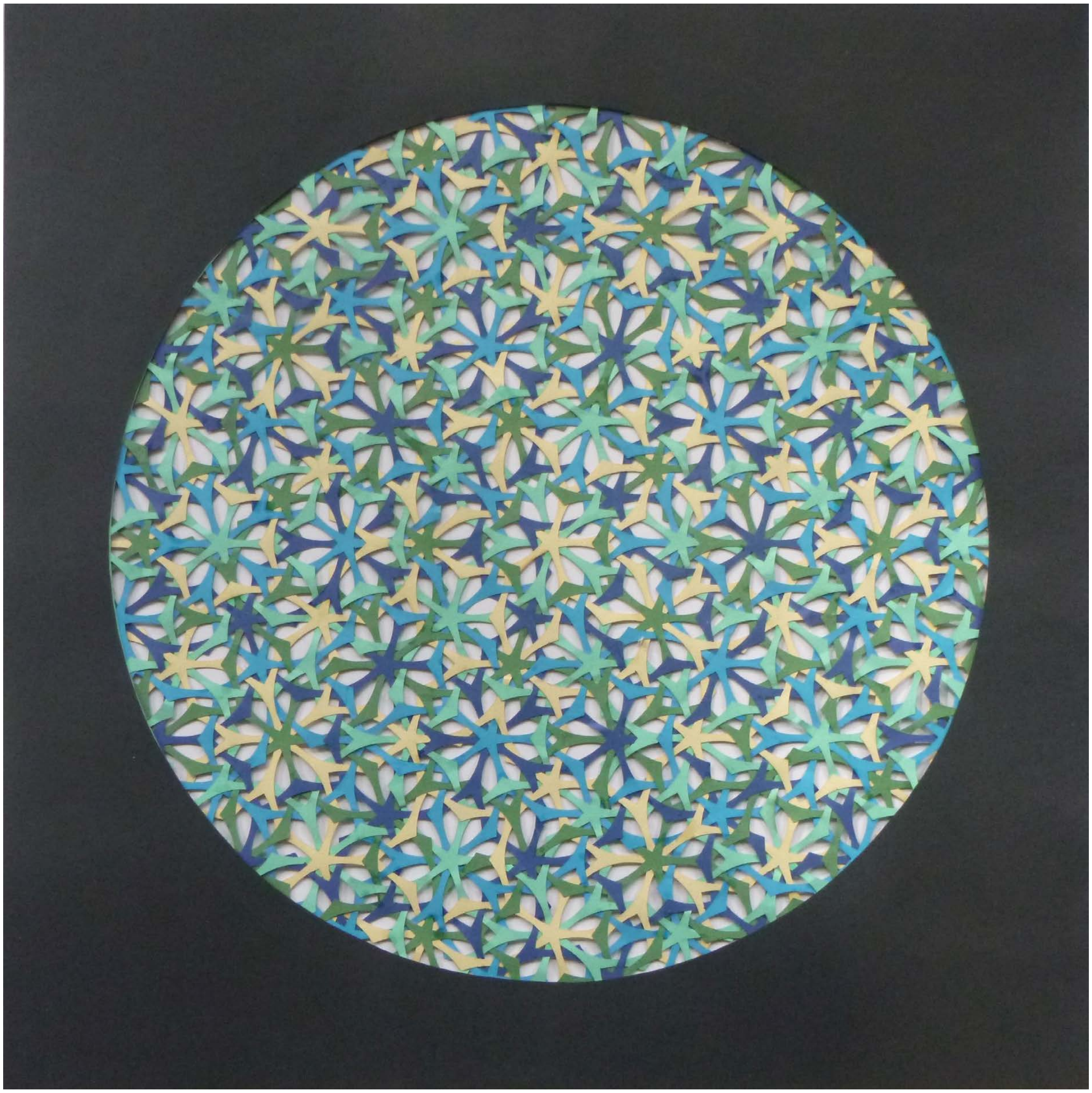

Figure 10: *The five linked starfish / ivy leaf / hexagon tilings.*

Summary and Conclusions

By a small modification in de Bruijn's cut and project method, every Penrose tiling is part of a set of five friend tilings. This shows a novel connection between tilings that is richer than simple shift–equivalence.

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