Creating Unicursal Curves Using Tiles

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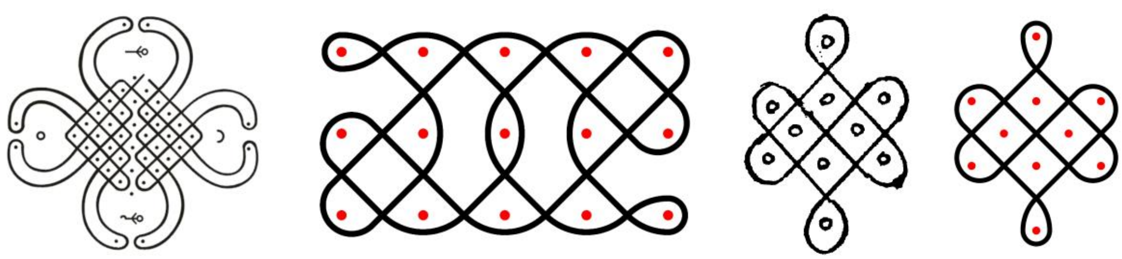
Abstract

This work discusses a tiling system used to create unicursal curves (drawn in a singular continuous motion), similar to those studied for their beauty, cultural importance, and mathematical content. The tile types and their designs are discussed, along with the analysis of unicursality. Methods for creating large images are presented, allowing the artist to exploit the power of digital computers and mitigate the limitations of drawing by hand. Several art images using this technique are included. While most of this work is done with square tiles, extensions to other tile shapes are considered.

Introduction

This work was inspired by that of the late Darrah Chavey [4]. At the 2013 Bridges conference, he presented works of wallpaper designs inspired by drawings called *sona*. Sona (plural, singular *lusona)* are often *unicursal* (drawn in a single continuous motion) and encompass a grid of dots. Figure 1(a) shows an example and Figure 1(b) reproduces an example illustration from Chavey's work, using the system discussed here. Geometrically, sona are similar to Indian *kolam* designs, an example of which is shown in Figure 1(c) [9] and reproduced in Figure 1(d). My interest is in digitally creating unicursal versions of such images, using tiles. The seminal piece on sona is probably Gerdes' book [5]. And, readers interested in creating images with tiles are also referred to Bosch's works [3].

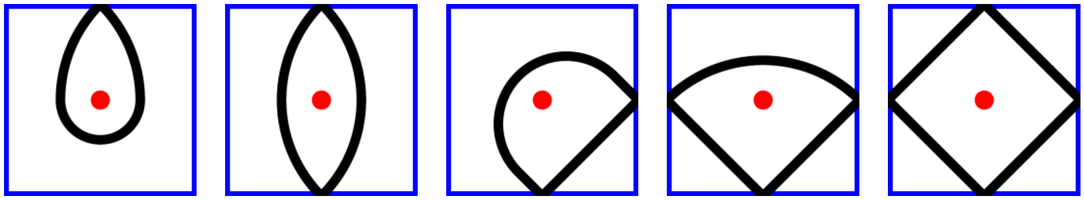


(a) (b) (c) (d)

Figure 1: *Examples sona and kolam drawings.*

Tile System

The tile system I developed is a compromise between approximating the creativity of the works done in indigenous cultures, with the precision and automation available with computers. Consequently, I worked with the program Ultra Fractal, in which the tiles were implemented as a fractal coloring formula. Figure 2 shows the five square tiles used, based on Chavey's work. The tile boundaries are shown in blue. Each tile has an optional dot in the center, to replicate the sense of the African and Indian drawings. I wanted the tiles to have maximal use, so the curve segments on the tiles enter and exit at the midpoint of the edge and at 45° angles to the edge, so that the curve would continue smoothly from tile to tile. The curved segments are made up of circular arcs, for ease in computation and for smoother images.



(a) (b) (c) (d) (e)

Figure 2: *Square tiles: (a) drop, (b) lens, (c) bump, (d) kite, (e) diamond.*

Analysis of Unicursality

While not required in general, I wanted my images to be unicursal, so I needed a way to analyze a tentative image and determine if the curve was indeed unicursal. For small images (such as those in Figure 1), it is easy enough to trace the image by hand. However, this can be quite a daunting task for more complex images and a computational approach was needed. To do this, I first needed access to the tiles' coordinates. Fortunately, Ultra Fractal allows saving the image's parameters as a text file. This was cleaned and loaded into a spreadsheet, where the type, location in the image, and rotation were captured for each tile. This list was read into a separate program, which "built" the curve as a series of arrays. An initial point was chosen and each segment on every tile was traced until the curve closed. If this curve used all of the segments on all of the tiles, then it was unicursal. If using all the segments created more than one curve, then it was not unicursal. In that case, the work began of adjusting the tile layout so that the new result would be unicursal, without losing the spirit of the original tiling. To facilitate this, I wrote the analysis program to output the path of the curves found, so that they could be plotted.

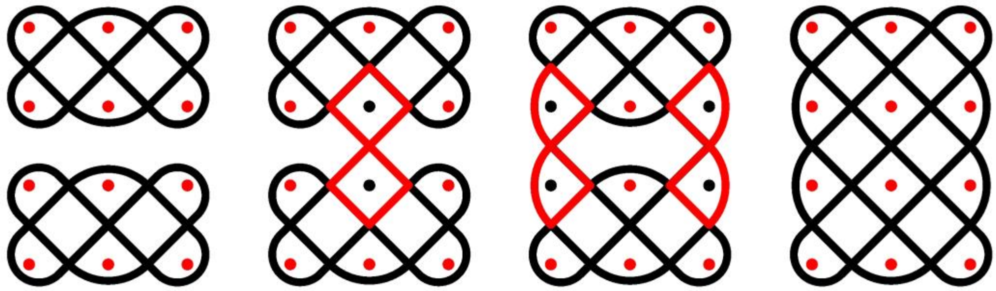
As an example, consider the curve in Figure 1(b), and begin in the upper left corner. This drop tile has one curve segment and the tile is rotated such that the segment begins and ends at the middle of the right side of the tile. Then, all 15 tiles and 35 other curve segments were interrogated to see which entry point matched this exit. In addition to matching the location, the curve must also preserve direction, and the general sense of the curve (in this case, traversing the upper left tile in a counterclockwise direction, coming out of the right side going upward). The routine would then identify the curved element of the kite tile as the next segment, and the process would continue. In practice, this image would produce two curves, one going counter-clockwise in the upper left corner and the other going clockwise, using the same segments but in the opposite order. These are just two versions of the same curve.

Creating Arbitrarily Large Curves

In addition to creating small curves such as that in Figure 1(b), I wanted to use this system to create large curves, suitable for display as intricate art pieces. Fortunately, there are some simple techniques that can be used to create larger curves from smaller pieces.

Tessellation

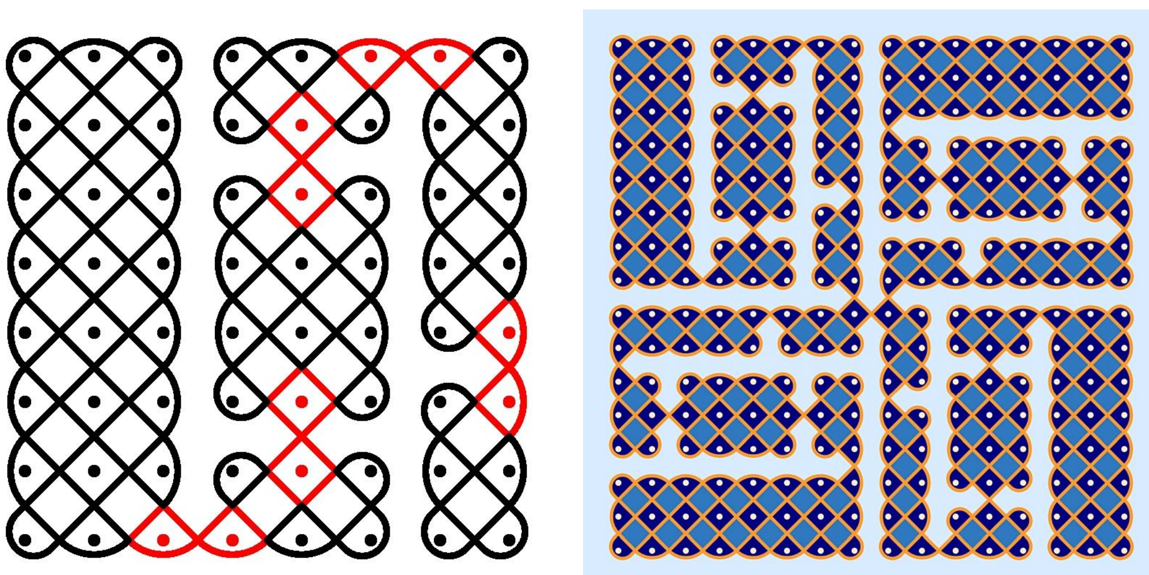
Figure 3(a) shows two unicursal curves. In Figure 3(b), two center kite tiles, one in each curve, have been changed to diamonds (red), joining the two curves into one. This new curve is unicursal, since both of the components were and since there is only one join point between the two. You can trace the top curve and when it leaves the top red diamond, it takes a "detour" through the bottom curve. Since the bottom curve by itself is unicursal, the detour ends up where it started and the tracing can continue in the top curve. Care must be exercised, however, when joining multiple curves. It's possible to break unicursality by having too many joins, as shown in Figure 3(c). Here, two joins create two individual curves. In this case, this can be fixed by making three joins (Figure 3(d)), which results in one unicursal curve, but with a different feel to it than the two original curves (Figure 3(a)).



(a) (b) (c) (d)

Figure 3: *Example showing how multiple unicursal curves can be joined into a larger curve: (a) two unicursal curves, (b) joined at one point resulting in one unicursal curve, (c) joined at two points and not unicursal, (d) joined at three points and unicursal*.

This technique can be continued indefinitely, so long as multiple connections don't break unicursality. For example, Figure 4(a) shows six curves joined together (with red tiles), similarly to those in Figure 3(b). The resulting large curve is also unicursal. You can trace the curve and see how it moves from one component to another, and back again. Figure 4(b) features a larger unicursal curve made up of four connected versions of the smaller curve. Also, the negative spaces have been colored for artistic interest.

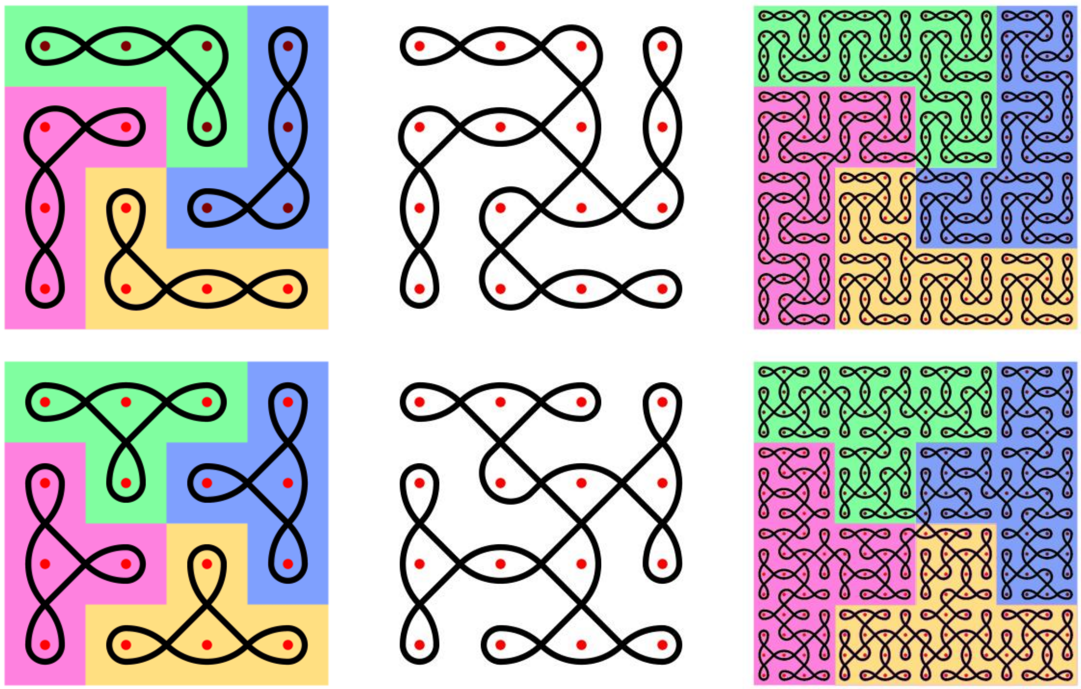


(a) (b)

Figure 4: *Example showing two levels of connection: (a) six elements connected to make one square unicursal curve, (b) four copies of the square curve connected to create a larger square unicursal curve.*

Quasi-fractal Curves

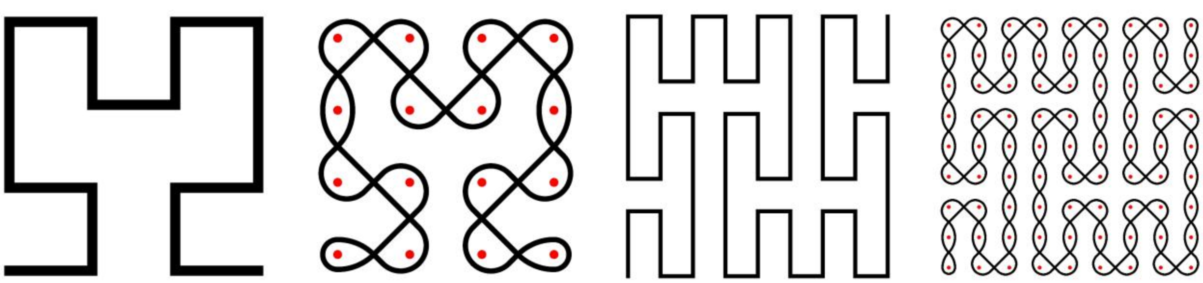
An extension of this idea is to create self-similar quasi-fractal curves, employing ideas from tessellation and geometric dissection. Figure 5(a) shows the L and T tetrominoes [10], how they are each made up of four squares, and how each tetromino can be represented by a four-tile unicursal curve. In both cases, four copies of the tetromino can be assembled into one larger square (Figure 5(b)). Then, four larger squares can be arranged into a large copy of the tetromino, and so on. Figure 5(c) shows the second iteration of both shapes, where four four-tile unicursal curves are connected to create a square-filling curve. Four of those are then connected to make a larger copy of the tetromino, and four of those are connected to create the final large square. Colored overlays show the larger tetrominoes and how they tessellate the square.



(a) (b) (c)

Figure 5: *Creating quasi-fractal unicursal curves using tetrominoes: (a) L and T tetrominoes with small curves, (b) combining the small curves into larger curves, (c) tessellating the square with larger curves.*

Creating unicursal curves based on Lindenmayer-system fractals [7] is particularly easy using a combination of lens, bump, and drop tiles. Figure 6(a) and 6(c) show low-iteration versions of the Hilbert curve and Peano curve fractals, respectively. In panels 6(b) and 6(d) are their unicursal representations. With this method, the lens tiles follow the flow of line segments, the bump tiles provide corners, and the drops terminate the lines and provide turnaround points for the curves. Expanding to larger curves is relatively straightforward, following the iteration process of the underlying fractals.

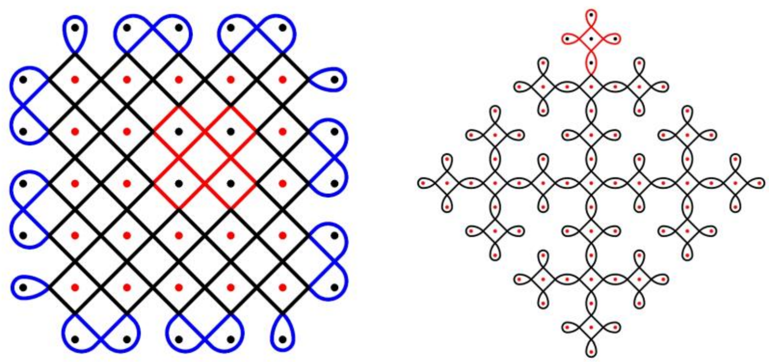


(a) (b) (c) (d)

Figure 6: *Unicursal representation of Hilbert and Peano fractal curves: (a) Hilbert curve, (b) unicursal version, (c) Peano curve, (d) unicursal version.*

Frieze and Wallpaper Patterns

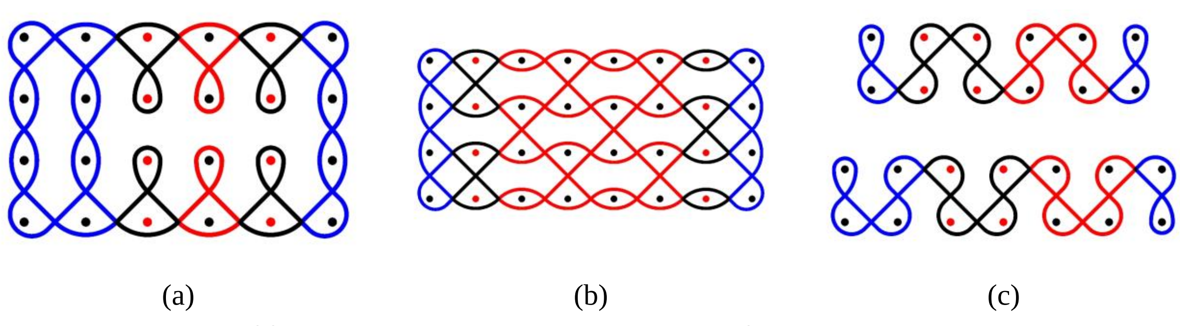
The small number of tiles suggests that unicursal curves will often have areas of symmetry and of pattern repetition. This suggests the question of whether or not a given pattern can be extended indefinitely and still create unicursal curves. Figure 7 shows two examples of square curves that can be extended. (In Figures 7 and 8, the red tiles indicate a fundamental block to be copied indefinitely and blue tiles are specific boundary conditions and are not to be copied.) Figure 7(a) shows a curve whose interior is entirely made up of diamond tiles. However, by using bump tiles in pairs and a single drop along the edges, the pattern is unicursal for an $n \times n$ square where n is odd. Care must be taken with the placement of the drop tiles, avoiding too much symmetry. Figure 7(b) shows a fractal-like star pattern. The fundamental group (red tiles at the top of the curve) is four drops around a diamond. However, when the stars are joined, the interior drops are changed to lenses and a diamond is added between each star.



(a) (b)

Figure 7: *Square curves that can be extended indefinitely in both dimensions: (a) square curve of diamonds with asymmetric edges, (b) star curve.*

Figure 8 shows examples of unicursal frieze patterns that can be extended indefinitely. Figure 8(a) is a row of drops bounded by kites. The blue lens and bump tiles establish the requisite symmetry-breaking and the red column shows the group to be copied. This curve is unicursal for any number of additional columns. Figure 8(b) is a pattern of two lenses and two kites, alternating with four kites. The repeating element is the red block of four columns, which relates to the frieze being four rows wide. If a wider band were used, then the repeating block would need to be longer. Figure 8(c) shows two strips each made up of bump tiles and a pair of terminating drops. In the upper strip, both drops are pointing the same direction and there is an even number of interior columns. The lower strip has one column of boundary bumps (blue) and the drops point in opposite directions. In both cases, the repeating block is four bump tiles in two columns.

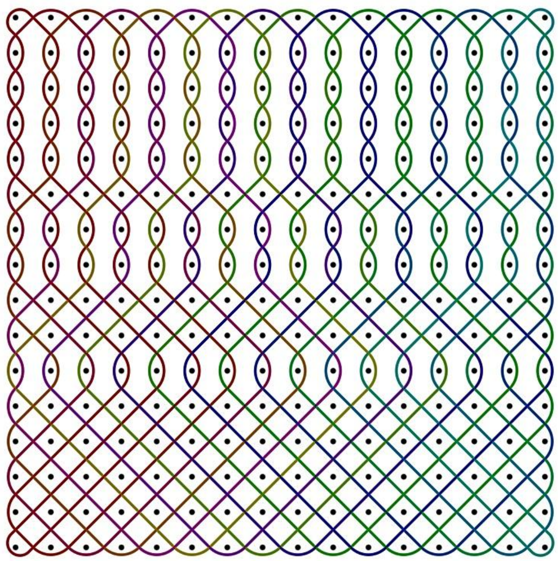


(a) (b) (c)

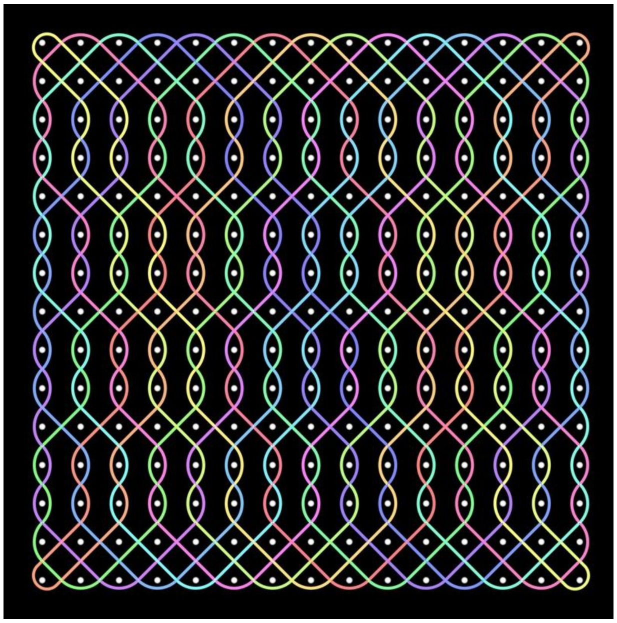
Figure 8: *Examples of frieze patterns that can be extended indefinitely in one dimension: (a) kites and drops, (b) kites and lenses, (c) bump tiles.*

Art Images

"Unicursal I" and "Unicursal II" in Figure 9 were early attempts at creating square unicursal images with symmetric tile layouts. Both employed a coloring method based on the curve segments, which facilitates tracing the curves. It was quite tedious to implement, but came out beautifully, in my opinion.



(a)

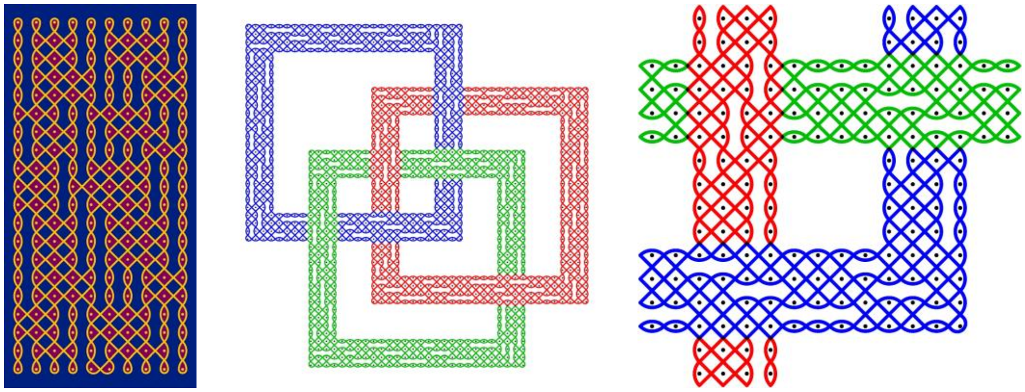


(b)

Figure 9: *Two square symmetric art images, both by the author: (a) "Unicursal I," (b) "Unicursal II."*

The image in Figure 10(a), "Tapestry," is suggestive of an ornamental wall hanging or a rug. Here, I used drops on the top and bottom edges to represent fringe and colored the curve gold to suggest gilded thread. The structure of the interior rows was a study in how to create sufficient asymmetry in the tile layout, such that a minimal amount of fixing was required to achieve unicursality. There are three types of rows; beginning from the bottom (above the drop fringe), there is one row of type 1, two of type 2, then two of type 3. From there, there are groups of types (three type 1, three type 2, etc.) progressing up the image. To determine the layout, I used a three-element Kolakoski sequence [1]. It is a self-referential sequence and begins: 1, 2, 2, 3, 3, 1, 1, 1, 2, 2, 2, 3, 1, 2, with each element counting the length of runs of 1s, 2s, or 3s. The numbers were codes for which type of tile group to use for each row. Since the sequence does not repeat, it was relatively easy to avoid symmetry and achieve a unicursal curve.

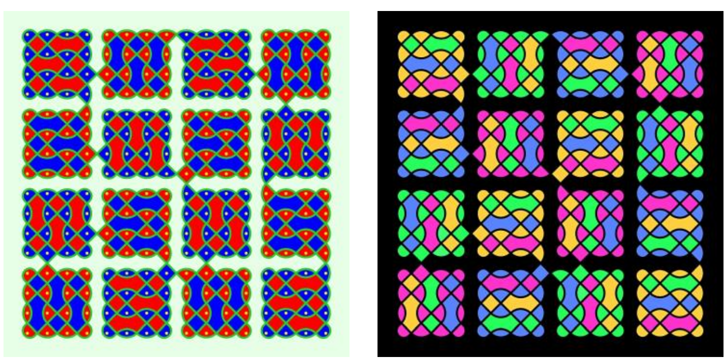
In a similar vein, "Square Borromean Rings" in Figure 10(b) is a unicursal representation of the famed rings [11]. While the colors suggest the linkages of the rings (red over green, green over blue, and blue over red), the overall image is one unicursal curve, without regard to color. Creating this was quite the challenge, as rectangular rings are often so symmetric that it's difficult to tweak them. Thus, I used the same sequence method as with "Tapestry" to vary to tile layouts along the rows and columns of the bands. Although there are over one thousand tiles in the image, only a handful of tweaks were necessary to reach unicursality. Knowing where to place them was the trick! There is one near the top right corner of the blue ring, where the visual gap is four tiles long instead of the maximum of three that the sequence provided. Figure 10(c) is an enlargement of the center, showing the detail.



(a) (b) (c)

Figure 10: *Art images, by the author, using a non-repeating sequence to achieve unicursality: (a) "Tapestry," (b) "Square Borromean Rings," (c) center of "Square Borromean Rings."*

The images shown thus far have just highlighted the curve and the dots at the tiles' centers. These dots are in homage to the lusona and kolam images made by cultures in Africa and India, respectively. However, the dots can be removed and the negative spaces highlighted for artistic interest. Figure 11 shows two versions of the same unicursal curve. It is composed of 16 rotated and flipped components, each joined to another in a sequential flow. While the curve and dots are shown in panel (a), the visual interest comes more from coloring in the spaces between curve segments. In addition, the component curves are laid out in a 4 × 4 Sudoku fashion, so that the same orientation (arrow directions and location of the red and blue diamonds) only occurs once in each row or column of four. The second version (panel (b)) dispenses with the dots, adds more colors, and allows the unicursal curve to be inferred in the background.

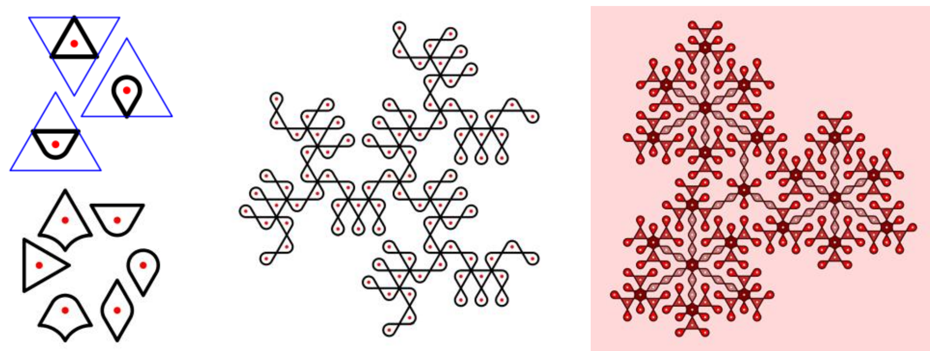


(a) (b)

Figure 11: *Two art images (by the author) highlighting the negative space rather than the curve: (a) "Four by Four I," (b) "Four by Four II."*

Extensions

The work discussed thus far is based on five square tiles. Future work could expand on this base by using more square tiles and/or tiles of different shapes. Additional square tiles could use different design criteria, for example, having more curve segments per tile or different edge constraints. Also, since equilateral triangles and regular hexagons also tile the plane, those shapes could be used as tiles. In Figure 12(a), I show three triangular (with blue boundaries) and six hexagonal tile designs (boundaries removed for clarity); panels (b) and (c) show two tilings from those sets. Or, perhaps including other shapes and supporting Archimedean tilings. I explored these ideas in previous work on generalizing Truchet tiles [6]. Tiles that are not regular polygons might also be considered, such as the thin and fat Penrose rhombi [8].



(a) (b) (c)

Figure 12: *Triangular and hexagonal tiles and example unicursal curves: (a) triangular grid tiles (upper) and hexagonal grid tiles (lower), (b) example triangular grid curve, (c) hexagonal grid image.*

Beyond tile sets, there is more work to be done in creating and analyzing specific tilings. For example, can random tilings be unicursal? And what kinds of tilings could be made in three or more dimensions? Finally, in what other ways can they be artistically rendered? Ahmed and Deussen's work [2] on halftoning images using lines may find some utility here.

Summary and Conclusions

In this paper, I have presented a system for creating unicursal curves using tiles. In a digital environment, this can allow for greater precision and larger scales than drawing the curves by hand. Several ways of combining and expanding base curves are shown, along with artistic considerations and works by the author. Finally, suggestions for future work are presented.

References

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