The Beauty of a Muqarnas Captured in Parameterized Software

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Abstract

Muqarnas are ornamental features of Islamic architecture. Their 2D floor plans resemble tessellations of squares, rhombuses and kites. However, these plans often oversimplify the spatial complexity, losing key details. The underlying grid may be octagonal, hexagonal or decagonal, but in the larger muqarnas, pentagonal or heptagonal shapes are added. A mathematician can prove that it is impossible to fit these shapes into these grids, but artists and designers know how to manipulate the muqarnas units so that pentagons do fit into an octagonal plan. The idea of describing a muqarnas in terms of pre-defined units works well for the smaller muqarnas, but fails for the larger. Having become stranded in the mathematical approach, Turkish architectural books offered a way out. Parametric units fit better into the design process. I automated the drawing process and today I can 3D print muqarnas designs.

Muqarnas as Works of Art

Have you ever stood before a mosque, gazed upward, and felt captivated by the intricate beauty above you? You may have been admiring a muqarnas, a breathtaking architectural ornament designed to astonish. Found in portal hoods, prayer niches (mihrab), beneath minaret balconies, muqarnas are masterpieces of spatial design. With their diverse materials, geometric complexity, and interplay of light and shadow, they have enchanted viewers for nearly a millennium. See Figure 1 for two examples in Iran and Turkiye.

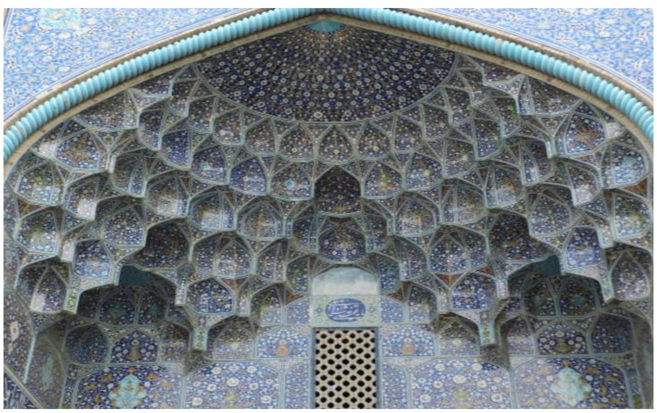



Figure 1: *(left) Isfahan Sheikh Lotfollah Mosque, (right) Istanbul Atik Valide Mosque.*

Senalp [7] offers a definition: “The muqarnas is a vaulting system based on the replication of units arranged in tiers, each of which supports another one corbeled on top of it.” Some emphasise the suspended nature, for example the rope-hung plasterwork in a wooden frame (Isfahan) or the hanging stalactite-like forms (Istanbul). Others highlight the stacked hewn stone that builds upward. In all cases, the regularity, symmetry, repetition, and layering stand out with the star at the apex, the origin point of the entire design.

Muqarnas appear across the Islamic world and beyond, in regions like Christian Armenia and Spain, a testament to the legacy of Islamic art. For centuries, mathematicians have been intrigued by muqarnas. Their story traces back to the 15th-century Iranian mathematician and astronomer al-Kashi, who computed the area of curved surfaces. Today, mathematicians dream of algorithms that can lift a two-dimensional tessellation into a coherent three-dimensional muqarnas form. It turns out that this kind of reverse engineering is as promising as it is disappointing.

Aim

My research aims to develop algorithms that enable the transformation of any two-dimensional tessellated floor plans into three-dimensional scale models by assigning standardized, pre-designed units. The goal is to incorporate this approach into rule-based software [4].

Lego-like Bricks

The replication of units gives rise to the idea of standardised, modular, pre-designed units. Could muqarnas units be assembled like Lego bricks? Often, the concept of pre-designed muqarnas units is promoted. My website has a long list of literature. Utrecht University organizes workshops for talented high school students, where they use 3D-printed, Lego-like bricks to assemble muqarnas. On one occasion, I was invited to lead a muqarnas workshop for primary school children. I decided to invest in a 3D printer and created seven boxes filled with hundreds of Lego-like bricks, enough for 28 students. Figure 4 shows an overview of these bricks. I use a color-coding system, which corresponds to the available 3D printer filament colors. Each specific muqarnas unit is printed in a unique color, allowing for clear visual differentiation. Blue is for the squares, rhombus are green, etc. Figure 2 shows some of these bricks. With these bricks, I could build small muqarnas. But when I attempted to create the larger portal muqarnas from Diyarbakır and Istanbul, I couldn't make them work. Why not? Was there something wrong with my bricks?

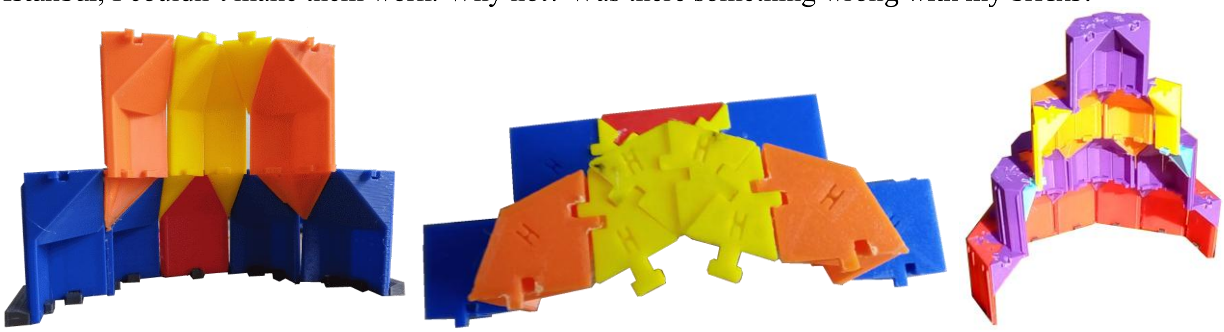
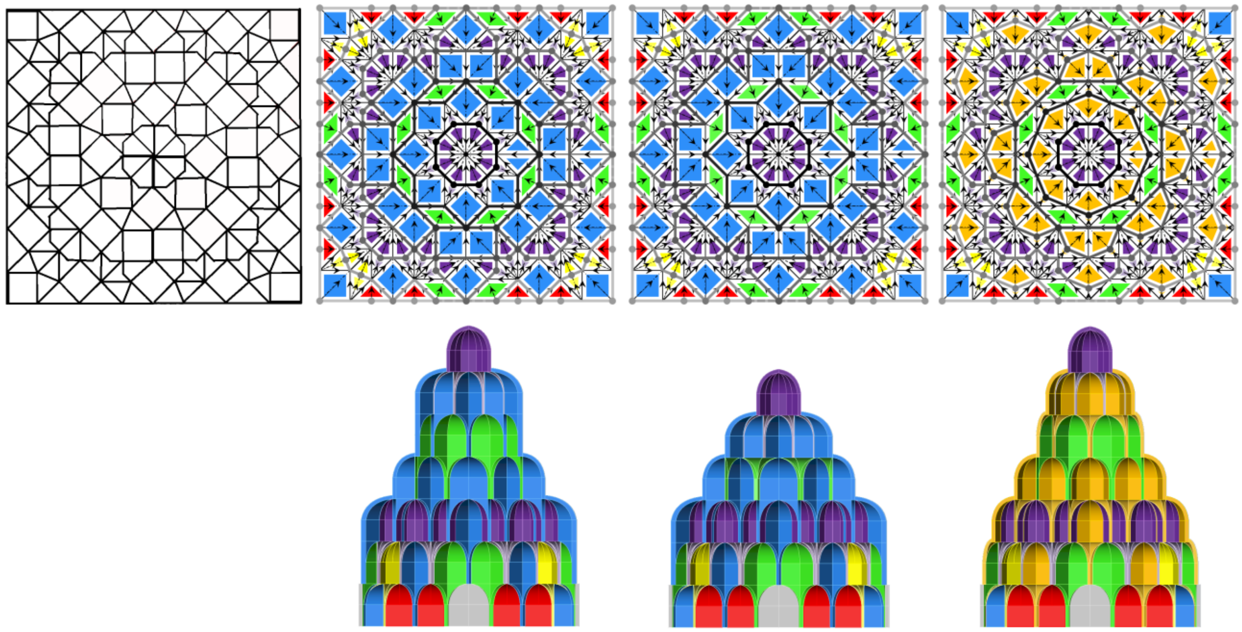


Figure 2: *Examples of muqarnas made of 3D-printed units.*

From 2D Floor Plan to 3D-printed Muqarnas: Bridging the Gap

Projecting a three-dimensional muqarnas onto a plane produces a distinctive tessellation. But such a projection inherently flattens space, raising a key question: can a 2D tessellation fully determine its 3D counterpart? The central challenge lies in elevating the tessellation: transforming a floor plan into a 3D structure by assigning pre-defined units. This reverse engineering lies at the heart of muqarnas design.

A slab discovered at Takht-i Sulayman, possibly representing a part of a muqarnas plan, fueled the idea that three-dimensional muqarnas could be generated from two-dimensional tessellation patterns. Harmsen [3] explored the feasibility of developing software to convert a 2D tessellation into a 3D structure using mathematical graph theory. Figure 3 shows a completion of the slab on the left, with three possible reconstructions on the right; just a few among many variations. The colors in the tessellation correspond to the unit types listed in Figure 4. The attempts in the middle employed true squares (blue A), but Figure 3(b) features an awkward turret, while Figure 3(c) has a compressed top resting on a base of three tiers with equal width. A more balanced and visually harmonious result was achieved in Figure 3(d) by replacing the squares with bisected units: orange jug (D) and biped (J). This example illustrates how the tiers of a muqarnas can be configured in multiple ways, each yielding a distinct spatial form. Consequently, the front views of these interpretations differ markedly. My software enables the visualization of such alternatives, allowing for iterative refinement. All I have to do is code sequences of digits, the letters of Figure 4. For figure 3(d) this code sequence would read C / D I D / J B J / D / D C J C J C / J B I D I F / E J E L A. It says that the top tier contains the purple (C), the second tier the orange (D) and purple (I), etc.



(a) (b) (c) (d)

Figure 3: *(a) Takht i Sulayman Tesselation, (b) Turret, (c) Compressed, (d) Balanced.*

Understanding the Grid

Shiro Takahashi's [8] website is a valuable resource for muqarnas plans and a tribute to Islamic geometric art. I reconstructed his designs using my Lego-like bricks with mixed success. Sometimes, the results resembled collapsed wedding cakes. What went wrong?

Through my travels across Türkiye and Iran, I have examined hundreds of muqarnas firsthand. At a glance, they appear symmetrical and orderly, but closer inspection tells another story. Subtle irregularities are the norm: tier heights vary, unit lengths are inconsistent, and symmetries are gently broken or manipulated. These pragmatic or intentional deviations enrich the visual experience. For instance, the portal muqarnas of the Atik Valide Mosque in Üsküdar, Istanbul (16^{th}-century), features pentagonal and hexagonal *püskül* embedded within an otherwise octagonal grid. See Figure 1. Mathematically, regular pentagons and hexagons cannot fit within such a grid; yet muqarnas defy these rules. Ödekan and Tuncer noted that muqarnas rarely conform to pure octagonal or hexagonal systems. The top may follow a different grid than the base, and in between lie transitional zones. Fieldwork shows that the repertoire of units goes well beyond those listed in Figure 4. Craftsmen, architects, and engineers worked with remarkable freedom, prioritizing the illusion of order over rigid geometry. Irregularity is not an exception. It is the norm. Perhaps the master builder had an entirely different understanding of grid, regularity, and symmetry altogether.

Literature in Turkish

Western scholarship has offered limited insight into the design logic of Turkish muqarnas. To bridge this gap, I had to delve into texts written in Turkish. Over the past fifty years, leading architects and scholars in Türkiye have produced substantial, insightful works on muqarnas. Among them, three Turkish professors stand out: Ödekan, Tuncer, and Uluengin. Their books deserve a wider international audience.

Ayla Ödekan [5] was a pioneering Turkish architectural historian specializing in Seljuk and Ottoman architecture. She initiated a classification of pre-Ottoman Anatolian portal hoods, identifying five distinct types. Her innovative method of arranging the octagonal or hexagonal grid using concentric circles, which she refers to as *geometric modules*, merits further exploration. See her floor plan in Figure 5.

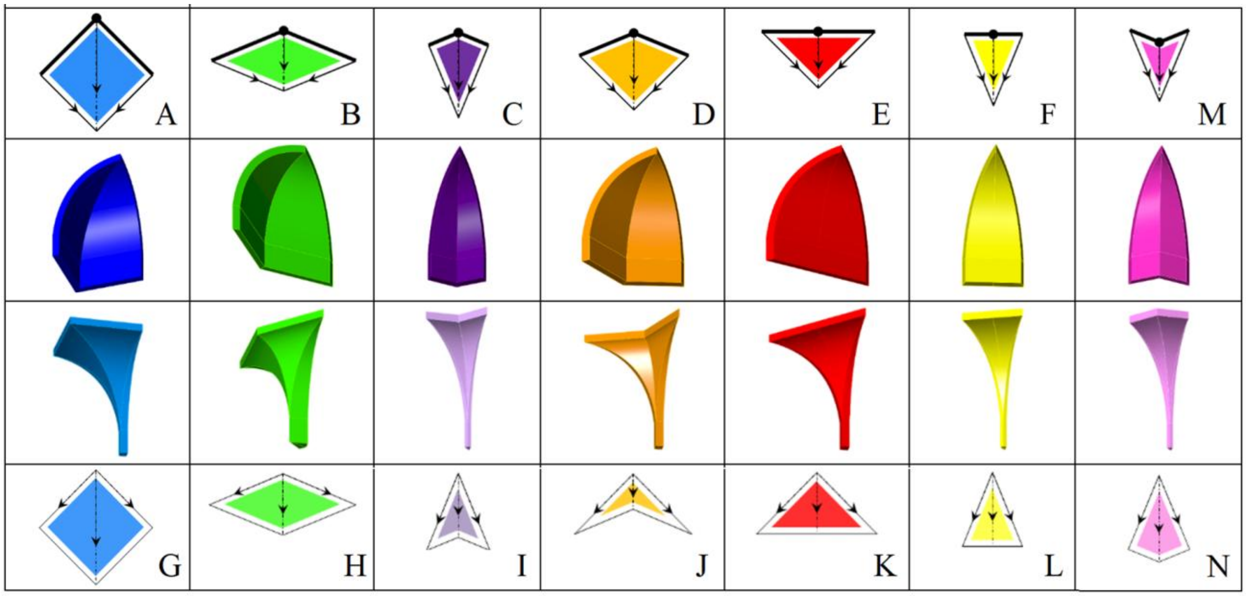


Figure 4: *Pre-designed units; top view and 3D view (including color coding and digit id's).*

Hüdai Sırrı Şenalp [7], an architect working within a family firm specializing in the construction and restoration of large mosque complexes (both modern and traditional), museums, and cultural complexes, brings a wealth of practical expertise to the design of muqarnas. As a scholar, Şenalp continues the academic legacy initiated by Ödekan, tracing the evolution of muqarnas from early Seljuk examples to the late Ottoman portal muqarnas seen in the work of Sinan.

Orhan Cezmi Tuncer [9], an architect, and professor, has written extensively on muqarnas across major Turkish cities such as Sivas, Ankara, Diyarbakır, and Kayseri. Remarkably, he drew these muqarnas freehand, with a level of precision rivaling that of photographs. I visited these cities and photographed their muqarnas firsthand. This fieldwork not only confirmed the accuracy of Tuncer's observations but also helped me develop a deeper understanding of regularity and irregularity in muqarnas compositions.

Mehmet Fatin Uluengin [10], the architect of Kocatepe Mosque in Ankara, has authored an outstanding book that guides readers through the mathematical and geometric aspects of drawing muqarnas. It also covers how to create molds for carving stones, determine the correct arrangement of the stones, and apply these techniques in the restoration of mosques, medresses, and hamams.

Grid

Understanding the underlying tessellation grid is essential to decoding muqarnas design. In Seljuk, Ottoman, and Armenian architecture, the dominant grid types are octagonal, decagonal, and hexagonal. In an octagonal grid, four lines of symmetry divide the muqarnas into four mirror-symmetrical parts. The fundamental angle here is 45°. The floor plan consists of a tessellation of squares and rhombuses. When a rhombus is bisected, it yields two isosceles triangles, each with a 45° apex angle and 67.5° base angles. For practical modeling, it is often useful to think in terms of 22.5° increments, that is half of the apex angle. A decagonal grid reveals five lines of symmetry radiating from the apex, with an apex angle of 36°. Here, all angles are multiples of 18°. In contrast, a hexagonal grid operates on an apex angle of 30°, with all angles expressed as multiples of 15°.

While these underlying grids can often be identified in the tessellation, they tend to appear only locally rather than governing the entire structure. A central grid for the entire tessellation is rare. Ödekan uses her geometric modules to link the local grids. Further research is needed to explore how the arrangement of these circles enhances our understanding of the interconnections between local grids. My website presents her floorplans, my interpretations and 3D animations, ready for 3D printing.

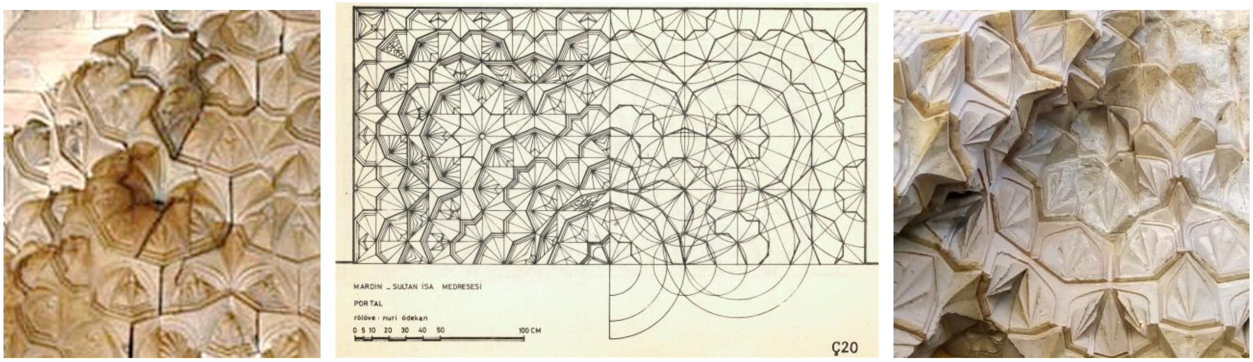


Figure 5: *(left) Mardin Sultan Isa: the "owls eye" has a small stalactite, (middle) Floorplan Ödekan, (right) Kayseri Sultan Han: the "owls eye" is a small dome-shaped niche.*

Different Views on Pre-designed Units

A key aspect of muqarnas analysis is identifying both the tiers and the elementary units that compose them. One method divides the muqarnas into horizontal tiers and then assigns units to each tier. However, this approach has limitations: some architectural elements span multiple tiers, with their top portion in one tier and base in another, obscuring their spatial coherence. Within this framework, a distinction is made between upper units (full units) and base units (intermediates). Typically, an upper unit rests on a base unit, but not always: upper units may also stand on other upper units, revealing a more flexible logic. Figure 4 shows upper units in the top row and base units in the bottom. Many more units exist as will become clear in a next section; a full list with three tables is on my website.

Sakkal [6], advocating a more systematic approach, catalogs all possible permutations where angles are multiples of 22.5°. He mapped out all possible configurations in the octagonal grid and identified over twenty pre-designed units. This raises an important question: do these theoretical units correspond to real-world examples? The answer is yes. Many of these configurations are found in Seljuk, Ottoman, and Armenian examples. In an unpublished study, Sakkal presents dozens of floor plans of Armenian muqarnas.

Ödekan, Tuncer, and Şenalp further emphasize the vertical articulation. In Anatolian Seljuk muqarnas, the units are classified into three categories: *yaprak* (upper), *kazayağı* (base), and *püskül*. The latter are those small hanging ornaments that typically appear at the center of stars. See Figure 1 and 5. Unlike the wide, radial stars common in Iranian designs, pre-Ottoman stars are characterized by yapraks at their apex, which converge to form a sharp, acute angle. Püsküls often take the place of smaller radial stars, and their horizontal cross-section is usually a regular polygon, but the surrounding star points are irregular, neither of equal size nor angle.

While mathematicians might argue that such shapes cannot be accommodated within a strict octagonal grid, artisans have demonstrated otherwise. The key lies in the flexibility of the yaprak in the horizontal plane. These units are not fixed: they can appear almost square, rhomboid, or kite-shaped, with angles that vary significantly. By relaxing the rigid constraint of angles as multiples of 22.5°, there is space, both literally and creatively, for the 72° pentagon to be integrated into the system.

In Ottoman muqarnas, Şenalp identifies two additional elements: *fitil*, a slimmer version of the kazayağı, and *badem*. The latter is an alternative to the yaprak. See Figure 6. Large Ottoman muqarnas often incorporate dome-like shapes and stalactite forms, resulting in a non-monotonous vertical progression of curves in 3D space. Two Seljuk examples are shown in Figure 5. Both evoke the early impression of "owl eyes": a small stalactite in Mardin, a small dome-shaped niche in Sultan Han. As a result of these forms, the tiers are no longer continuous curves in the horizontal plane. While the stalactites emphasize the "hanging" quality of muqarnas, the overall structure still reads as a rising pile of stones. Since the symmetry radiates from the apex, I number the top tier as one, the tier beneath as two, and so on downward.

Figure 6: *(left) Pre-Ottoman yaprak is the upper unit and kazayağı its base, (middle) Ottoman kazayağı is the base of a badem, (right) Ottoman fitil is the base of a badem.*

Considering these insights, I made several key decisions for my project. My Excel Visual Basic software adopts the tier-by-tier approach, coding muqarnas in terms of upper and base units, as this was how I began. Each unit is defined parametrically in a worksheet. Every shape is treated as a kite; even rhombuses and squares fall within this definition. By specifying just two angles, the angle at the back and the angle at the front, a kite shape emerges. Additional parameters include curvature and dimensions, which control the unit's appearance in three-dimensional space. Following Sakkal, the Excel software merges upper and base parts across tiers into single objects, which are automatically detected and grouped. Encouraged by Alaçam, the Grasshopper application uses a single parametric object as the core building block governing every muqarnas shape. Finally, the resulting design is exported from Excel via Rhino as an STL file for 3D printing, allowing me to inspect a tangible miniature of the digital model.

Do We Need Pre-designed Units?

Until now, I have focused on pre-designed units with fixed angles. Yet, as discussed earlier, Turkish elements like the yaprak and badem often defy strict angular constraints, approximating angles like 30°, 36°, or 45°. This suggests that rigid unit libraries may not be the best choice.

Modern computer graphics offer exciting new possibilities. Historical techniques, such as Al-Kashi's use of squares and circles, can now be extended with advanced geometries like rectangles, ellipses, and smooth curves such as b-splines, made possible by modern drawing software. Future research will explore how parametric modelling might replace pre-designed units, enabling greater flexibility and creative freedom. For instance, entirely new forms could emerge if the underlying grid were based on ellipses.

Analysis of a Muqarnas: Three Examples

Sakkal identified many more unit types than the combinations of upper and base listed in Figure 4, and so did I. In this section, I analyze specific muqarnas examples to demonstrate the existence of these additional units. While only a few examples can be presented here, a comprehensive overview is available on my website. Figure 7 shows a muqarnas in Diyarbakır. At the left, the tiers are numbered from the top.

In Harmsen's set of twelve units, the units needed for the top tier of the Kasım Paşa muqarnas are missing. Turkish muqarnas often feature sharply carved, pointed elements, which led to the addition of two indispensable units (N) and (M) shown in the right column of Figure 4. These two forms appear frequently in Seljuk and Ottoman examples, often at the apex, but also in the lower tiers. Figure 7 shows an example.

Some shapes in Figure 7 appear in multiple sizes. The orange unit (D) comes in three versions: small in the second tier, large in the third, and regular in the fourth and sixth tiers. The small (D), an upper unit, sits atop a large (I), forming a kite-shaped composite. The large (D) has no dedicated base; it rests directly on a regular (D) below. Despite these size variations, all elements still fit within a single octagonal grid.

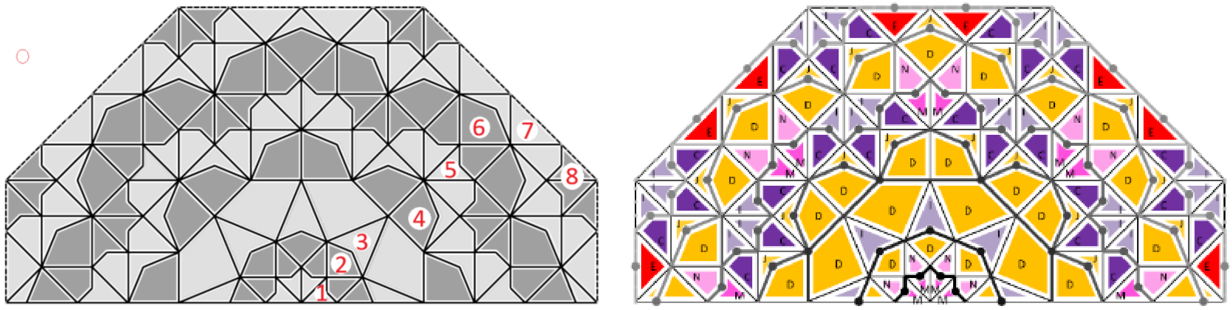


Figure 7: *Diyarbakır Kasim Padisah (left) floor plan with tiers, (right) My assignment of units.*

At the top, small upper (M) and base (N) form a rhombus. Regular (M) and (N) reappear in tiers five and six, alternating with upper (C) and base (I) to form another rhombus, while upper (D) and base (J) combine to create a square.

The Seljuk portal muqarnas of Afyon Çay Taş Medrese (Figure 8) features yaprak and kazayağı, along with three striking vertical panels adorned with lion motifs and calligraphy. Harmsen proposed using mathematical graphs. This muqarnas would have been an ideal test case. Could she handle it? I incorporated edge-to-edge units into my coding system, enabling my software to decode its structure and generate a 3D-printable miniature. Note that rhombus (B) never rests on a base unit, but atop another upper unit. Also note the decorative differences between yaprak (B) and (D).

Figure 8: *Afyon Cay Tas Medrese (left) Portal muqarnas with yaprak, kazayağı, and vertical panels, (right) Exploded front view with my assignment including new vertical units.*

Figure 9 shows the prayer niche of the Marmara University Theology Faculty Mosque in Istanbul, a contemporary structure designed in the Ottoman style. Despite its modern construction, the muqarnas has rare units near the top, echoing historical precedents. Similar configurations can be observed in the 13th-century Sultan Han located between Konya and Aksaray. In this arrangement, a rhombus is divided into an upper unit (T) and a base unit (U), which meet at a 90° angle and are flanked by half units (I).

The example in Figure 5 illustrates that it is impossible to determine from a tessellation alone whether a tier extends upward (as in a small dome) or downward (as in a small stalactite). These 14th-century muqarnas are relatively simple compared to the more elaborate Ottoman designs, which feature impressive “owl’s eyes,” stalactites, pentagons, and more. This comparison underscores the importance of closely examining the physical structure of muqarnas, rather than relying solely on tessellation patterns. Alaçam [1], Dinçer [2], and others advocate for the use of photogrammetry, as it can yield more accurate results than methods based on assumed grids that may not actually exist.

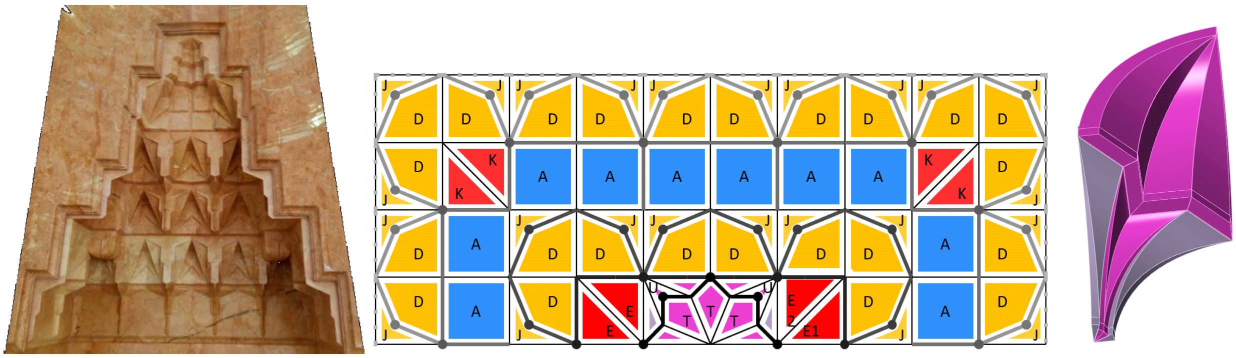


Figure 9: *(left) Prayer niche, (middle) Floorplan with assignments, (right) Units T and U are in the top.*

Conclusion

The concept of pre-designed units within a system of horizontal tiers is undeniably appealing. However, as structures grow larger, assigning units to the tessellation becomes increasingly cumbersome. This leads to a proliferation of unit types and offers no clear solution for incorporating heptagonal stars into an octagonal grid. Exploring traditional concepts such as yaprak, kazayağı, badem, and fitil proved far more effective than rigidly categorizing fixed-angle units as upper or base. This shift in perspective paved the way for parametric design, which, in turn, simplified software development. Parametric units allow for greater flexibility, making it possible to incorporate stalactites with polygonal bases.

For a deeper understanding of muqarnas, I encourage readers to visit Türkiye and engage with the foundational works of Ödekan, Tuncer, and Uluengin. AI can assist in reading Turkish texts. I also recommend completing the full research cycle from interpreting a tessellation to 3D printing a miniature muqarnas. A physical model conveys understanding in a way a digital rendering simply cannot.

Acknowledgments

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A more extensive list of references, including online sources, can be found on my website: www.henkhietbrink.nl/muqarnas.htm. Here, I would like to mention influential Turkish authors.

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