Transforming Polygonal Objects into Multi-Functional Furniture from a Given Net

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Abstract

One of the primary challenges in interior design is the efficient organization of small spaces. While the spatial constraints are typically predetermined, the selection of furniture must be approached with careful consideration by the client to maximize both functionality and aesthetics. To address this issue, we suggest multi-functional furniture solutions, which raise both applied and theoretical mathematical questions regarding the folding and unfolding of polygonal objects within a given spatial configuration. As a case study, we suggest multi- functional furniture/objects, each serving a distinct purpose, that can be folded into a plane.

Introduction

The main goal of interior design is to create a functional and aesthetic environment that meets the needs of the client while taking into account various spatial constraints. These constraints include the available space, the intended use (such as commercial, public, or residential), and the necessary furnishings. In recent years, urban development and population growth have posed new challenges for interior designers, especially in optimizing limited spaces. Designers must address spatial division to accommodate essential functional areas (such as kitchens and bathrooms) while incorporating furniture that aligns with both practicality and aesthetics. While functional requirements are predetermined, furniture selection is usually determined by the tenant. It should be noted that furniture occupies approximately 40–50% of the total floor area (for more details, see [7, 12]). Some furniture, such as beds, are essential for daily use, while others—such as additional chairs, coffee tables, and stools—may be used less frequently. Given these constraints, it is necessary to optimize floor space, reduce the number of non-essential furniture items, or design multifunctional furniture that can be used for different functions at different times. This leads us to the question of whether certain furniture can be defined as modular furniture, and even combined into a single unit that can be used for a different function at a time or even folded flat for storage? Such modular furniture should be efficient, aesthetically appealing, functionally practical and easily modifiable. In addition, clear instructions should be provided to ensure user-friendly operation.

This process naturally leads us to the concept of origami, a transformation from a flat plane into a three-dimensional object, where mathematics plays a fundamental role. The potential of origami-inspired furniture has already been discussed in [1, 8], where the primary objective is to utilize the principles of origami to allow a objects to fold into a compact, planar form.

We aim to apply origami principles to the design of multi-furniture, where the mathematical formulation is as follows: Given polygonal objects A and B, can we define a folding transformation f such that $f(A) = B$? Furthermore, does the inverse transformation $f^{-1}(B) = A$ exist? If so, what is the minimal number of steps required for this transformation in terms of folding and unfolding operations? The transformation must satisfy the conditions that object A is completely converted into B without residual parts, and no tearing or gluing is permitted.

Unfolding a polyhedron, when possible, leads to different non-isomorphic planar nets, for example, a cube has 11 distinct non-isomorphic unfoldings, while the dodecahedron has 43380, for further details regarding the number of polyhedron unfoldings, see [4].

A similar approach to origami and mathematical furniture has been discussed in [6] using Euclidean geometry, where furniture must arise from a single piece of material and without addition.

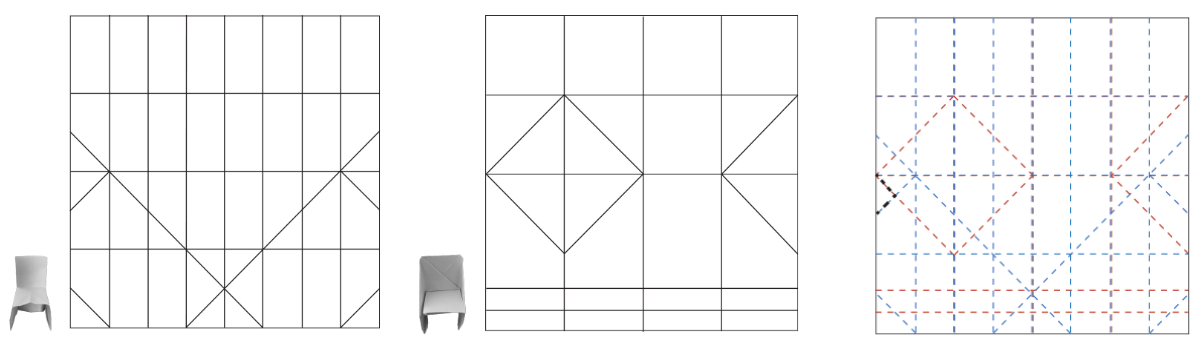
From our point of view, the above mathematical questions naturally led us to Erik D. Demaine, a pioneer in the development of folding algorithms across various contexts. His contributions have highlighted numerous open problems in this field, as discussed in [2, 3]. Since folding algorithms must ultimately be applied to physical objects, it is essential that our algorithms optimize factors such as material usage, the number of tiles, and user-specific constraints, as in [5, 9].

Lastly, our mathematical aim is to develop a generalized algorithmic tool that accepts a given net. The algorithm will generate all possible folding objects and determine whether a closed surface is contained in this object. The objects that fulfill this term will be introduced to the designers and are the candidates to be part of the multi-functional designed object.

Our Results

Mathematical Formulation and Experiments

While designers focus on defining the physical object, the mathematic team intend to develop folding and unfolding algorithms necessary for achieving the designers' objectives. Our primary objective is to define multiple foldable objects from a given net, ensuring that, with appropriate folding instructions, an object A is transformed into B. This transformation must satisfy the constraints that A is fully converted into B without any residual fragments, and no tearing or gluing is permitted in the original net.



(a) **(b)** **(c)**

Figure 1: *The left and middle images depict two paper nets with the corresponding folding instructions, each of which leads to a distinct object (in this case, different chairs). On the right side, dashed lines are added to illustrate the existence of a shared net, which is defined by the bold triangle.*

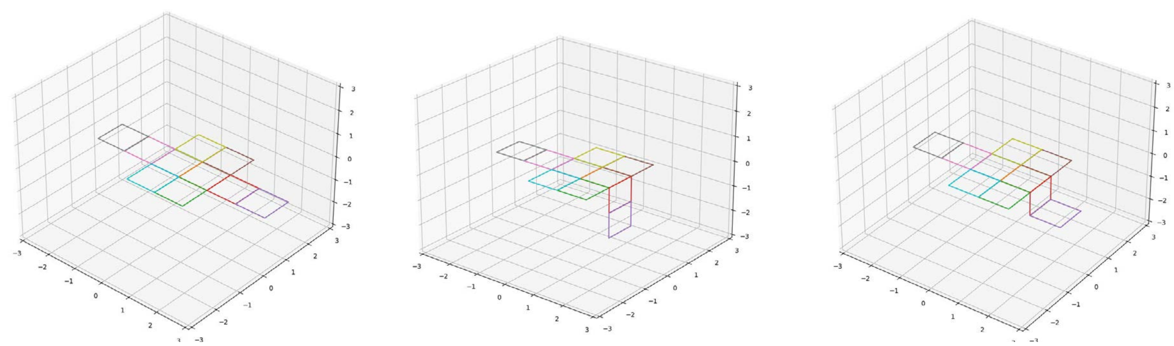
The difficulty of these questions led us to define and explore a case study based on origami ideas. In Fig. 1a and Fig. 1b, we produce two different types of origami chairs from given nets, with different folding instructions. In Fig. 1c we made a superposition of the two different nets on one plane. As can be seen, there exists a tiling which can construct both different objects (the bold triangle). The experiment concludes that there is a transformation from object A to object B if both objects can be derived from the same planar net.

In origami, the folding process starts with a plane, and the result is a $3D$ object which indeed with specific constraints meets our needs.

We focus on discrete folding nets, which we will define as a net that is made up of polygons, and the folding operations can only be made along edges. To simplify our problem, we will focus on a finite square net λ that can be folded along the edges alone, at angles of $\theta = \{0°, 90°, 180°, 270°\}$. If we enumerate the faces of the net arbitrarily, folding the i-th face in an angle $j_i \in \theta$, will be denoted by $T_i^{j_i}$ ($T_i^{j_i}$ can defined on any $3D$ surface obtained by folding λ and not only on a $2D$ net) and a $3D$ object defined by a sequence of folding will be written as:

$$T(\lambda, \vec{j}) := T_n^{j_n} \circ T_{n-1}^{j_{n-1}} \circ \ldots \circ T_1^{j_1}(\lambda) \ ,$$

where $\vec{j} = (j_1 \ldots j_n)$ and n is the number of folds. For the sake of simplicity, each face can be folded regardless of the current shape of the net; In other words, the order of the folding does not matter, and the interior of two faces or more can intersect with each other. Fig. 2 is an example of legal folding operations where we numbered the red square as 1 and the purple square as 2, etc.



(a) **(b)** **(c)**

Figure 2: *In (a) the initial net. (b) Apply $T_1^{90°}$ - the first folding in 90°, which leads to the red and the purple squares folding. (c) Apply $T_2^{270°}$ - another rotation with respect to the purple square.*

Since the number of possible objects for a given λ is exponential to the number of faces, we would like to filter specific shapes that fit the designers' needs. After a discussion, we agreed that the shapes of interest generally contained a closed polyhedron.

For this purpose we propose Algorithm 1. it is our brute-force algorithm where the square net is organized on a heap structure as follows: a random square has been chosen to be the root of the heap and each layer i is defined by the neighbors of the $i-1$ layer (not including the squares in the previous layers, which defined a distance from the root). Lastly, we will enumerate each square in the net in post order for each $1 \leq i \leq n$ by $\{H[1], \ldots H[n]\}$ ($H[n]$ is the root), i.e., the first fold will be respective to $H[1]$.

To identify closed polyhedra, we came up with the following saving condition in lines 8-9, which is based on finding all squares that have four neighbors of degree four, which will be denoted as X.

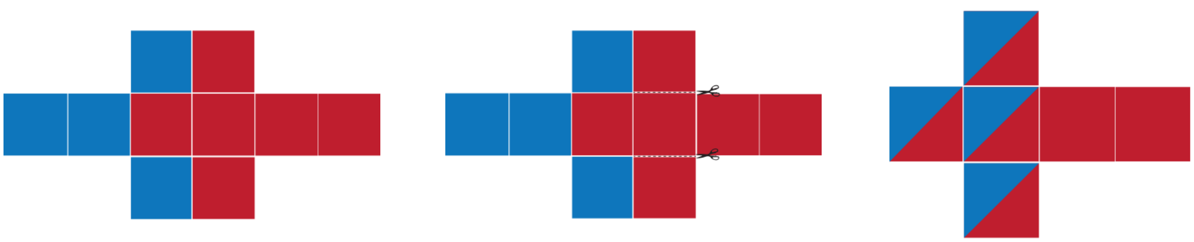
The possible output of the net was presented to the team designers, who examined the net and saw from their perspective that this net could be transformed into a number of different pieces of furniture as it contained both a cube net and a cuboid net. After a long discussion, three items were selected to fill this modularity: a cube, a cuboid, and a chair. From a design perspective, the cube resembles a stool (which requires careful design). Next, we selected the cuboid, which, in the designers' view, resembles a coffee table, whose corresponding nets can be obtained by adding four faceted squares to the selected cube net, which still retains the properties of the cube net during folding.

Notice that unfolding the cube leads to 11 different nets options, while unfolding the cuboid yields 54 nets and the chair net must be contained within the previous nets. In Fig. 3a, the red squares represent a chosen

Algorithm 1: Folding algorithm to a square net.

Result: The set of possible surfaces
Data: Given the post order array $\pi := \{H[1], \ldots H[n]\}$, which represents the square net.
Define angle(H_i) as the angle between H_i and H_{i-1}.
Define Fold(H_i) as the number of $90°$ folds respective to its parent in the heap.
Define X as the number of squares where each has four neighbors.

```
i = 1
S = ∅
while i < n do
    if angle(H_i) == 0 then
        i ← i + 1
    else
        π ← Fold(H_i)
        i ← 1
        if X > 2 ∧ (X mod 4 = 2) then
            S ← S ∪ π
        end
    end
end
return S
```



(a) **(b)** **(c)**

Figure 3: *The initial net is the red net which is respective to a cube, where by adding the blue square tiles leads to a cuboid.*

unfolding of a cube, and adding the blue squares results in the unfolding net of a cuboid. In Fig. 3b, we perform two cuts, which, through reflection, lead to Fig. 3c, representing the unfolding of the cube where the two-colored square consists of two layers (blue and red). A short animation demonstrating how each object is obtained from the given net with their respective folding instructions can be found in [10].

In Fig. 4, the objects explored are derived from the given net. In Fig. 4a, the cube will be approximated as a stool, while Fig. 4b, the chair we produced incorporates structural strength through triangular-shaped legs. Lastly, Fig. 4c, the cuboid will be approximated as a coffee table designed for storage.

The Design Process

The initial 'naive' net presented lacked a design approach. We analyzed this configuration to assess whether an alternative net could be developed to produce all three pieces of furniture while adhering to key principles, such as structural strength.

After selecting the net, we applied optimization principal to identify parts that did not affect the structural

(a) (b) (c) (d)

Figure 4: *A naive approach to multi-object respective to the net in Fig. 3, whereby the right folding instructions lead to three objects: a cube, a chair and a cuboid. Lastly, we found that net (d), which is another net that leads to the objects, improves the design process.*

(a) (b) (c) (d)

Figure 5: *Our real objects after the design process: stool, chair and coffee table. Made of cardboard, faux leather, and magnets, which increase the structure of each object.*

strength and could be subtracted or redesigned. For these areas, we made several modifications using various techniques, such as notching, subtraction, and bending. Following these modifications, the three pieces of furniture could still be built while maintaining the guiding principles with minimal excess material.

Several challenges were encountered in building the physical model: Due to the thickness of the material, we had to determine the exact spacing between the tiles to achieve a 90° folding angle. We also realized that certain tiles, such as the seat of the chair, could not be removed or modified. Therefore, we carefully selected tiles where material could be subtracted from the overall shape, all this to ensure that the furniture is both functional and comfortable. From a design perspective, we decided to subtract material in a way that would add value to the final product. Instead of removing material symmetrically or in straight lines, we designed subtractions that would enhance the visual appearance, compare Fig. 4d, which is an adjustment of the algorithm output and the designed result in Fig. 5a.

Using the subtraction technique, we realized that we could add aesthetic value expressed through practical application. For example, instead of two parts sitting on top of each other and creating an overlap of material, we removed one part and filled the space created by the other. This approach allowed us to maintain a single thickness of material without duplication, which also contributed to better storage of the product.

The resulting designed furniture is shown in Fig. 5, for an animation of Fig. 5 see [11]. Although the functionality of the furniture remains unchanged, the design process has introduced notable improvements. For example, the chair is now equipped with handles and the coffee table features a design with legroom for comfort and usability.

Conclusions and Future Work

The mathematical concept of folding a plane into a polyhedron, and vice versa, inspired our approach to designing multifunctional furniture for small spaces, which can serve various purposes and can be folded to a plane for storage. In this work, we present the initial process of using folding algorithms applied to multi-functional, well-designed objects. We show how to translate this theoretical approach into practical applications for real multifunctional furniture, taking into account appropriate materials, weight considerations, comfort, and other essential factors.

In future work, we intend to develop an algorithm in the other direction: Given a collection of $3D$ objects, the algorithm identifies the corresponding unfolding net for each object and determines whether there is a common net that unfolds for all objects in the collection. The designers then assess whether this unfolded net meets their requirements in terms of structural strength, functionality, aesthetics, and other criteria.

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