Duality: Connected Structures

Rinus Roelofs

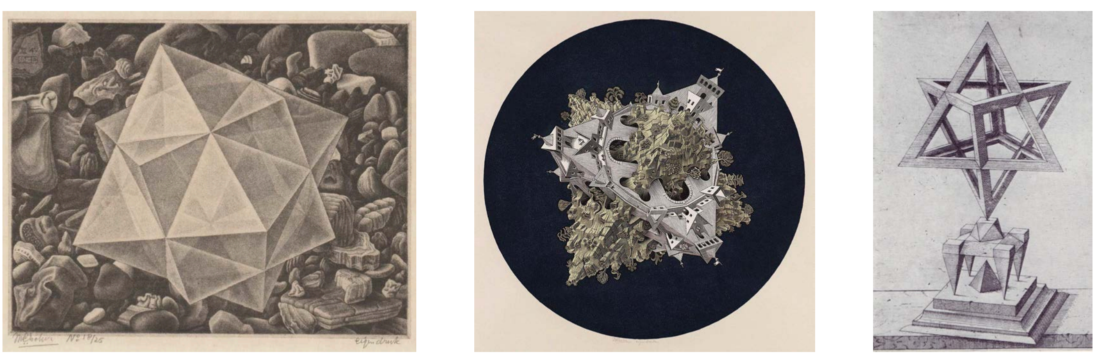
Hengelo, the Netherlands; rinus@rinusroelofs.nl

Abstract

Duality is a concept in mathematics that underlies "Dubbel Planetoïde" (1949) and "Kristal" (1947), two prints by M.C. Escher. In these prints, we see a perfect combination of two polyhedra that apparently can hardly do without each other. In Escher's print "Kristal," both the cube and the octahedron are drawn as solid shapes. In this paper, I have now opted for an extra reversal: instead of both shapes being solid, I have replaced one of the two shapes with the skeleton of the shape. So, the cube is replaced with a figure that you get by extruding all the edges of the cube to the center of the cube; a figure consisting of twelve triangles. The result is that the combination has now become an easy to assemble construction.

Introduction

The concept of duality has inspired several artists and among them M.C. Escher maybe the most well-known. He illustrated this phenomenon most beautifully in prints like "Crystal," Figure 1(a), and "Double Asteroid," Figure 1(b) [2]. Renaissance artists studied these kinds of constructions. As an example, we see a drawing made by Wendel Jamnitzer in Figure 1(c) [1]. When we now compare Jamnitzer's drawing with Escher's print "Double Planetoid," we see that in Escher's drawing the faces of the tetrahedra are shown, while Jamnitzer only drew the edges. That brought me to the idea to study another way of combining the two dual objects; a way that would it make easier to make real physical constructions of the dual combinations. I found a method that can be described as "dual-extrude-construction."



(a) (b) (c)

Figure 1: *Duals in art: (a) Cube-Octahedron, (b) Tetrahedron-Tetrahedron, M.C. Escher, (c) Tetrahedron-Tetrahedron, Jamnitzer.*

Dual Extrude

If we look at the dual pair of the entwined tetrahedra, as in Figure 1(b) and (c), we can make a new structure by building up one of the tetrahedra with four triangular faces and the other by six extrusions from the six edges of the second tetrahedron to the center of this tetrahedron.

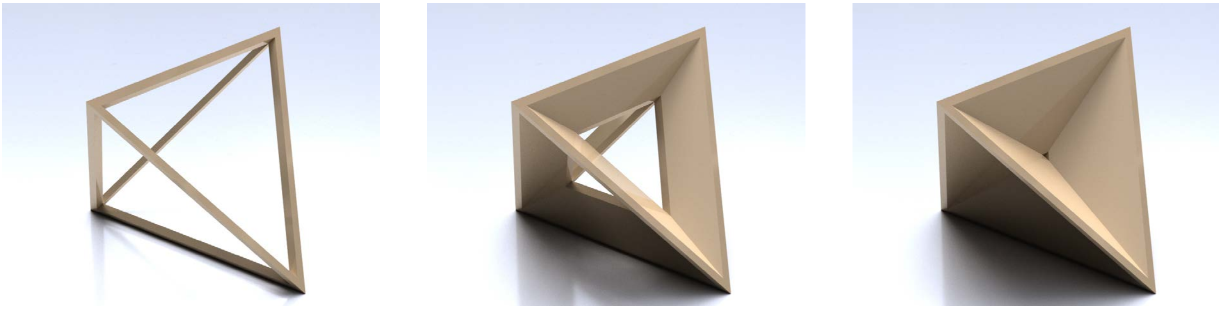
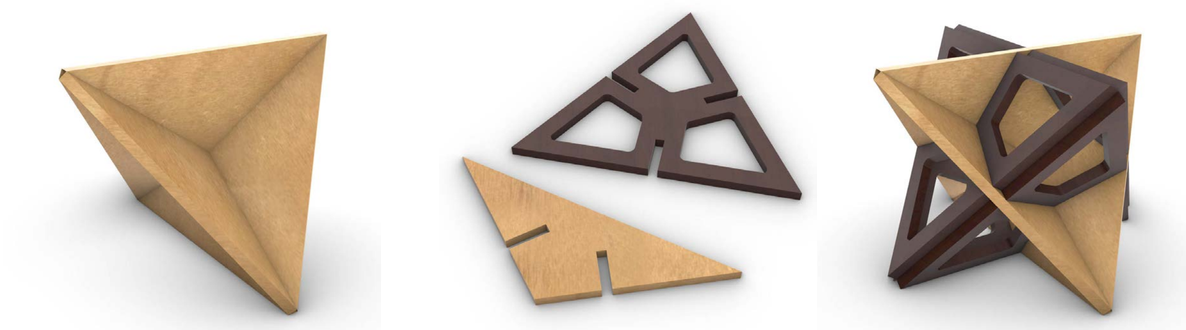


Figure 2: *Extrusion of the six edges of the second tetrahedron to its center.*

Slide Together

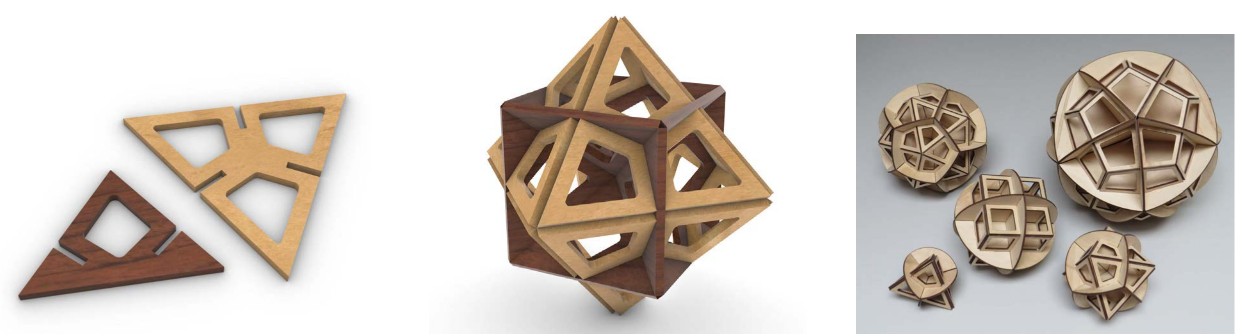
For the overall construction, we now have two sets of triangles: four equilateral triangles and six extruded triangles. After making half-way slots in each of them, as in Figure 3(b), we can slide everything together to get the construction shown in Figure 3(c). The connection of two different types of parts is orthogonal. This means that the slots in the parts are easy to make and the connection is then quite simple. I made holes in the equilateral triangular faces to get a good view of the inside of the construction.



(a) (b) (c)

Figure 3: *Dual-extrude construction of the tetrahedron. (a) extrusions, (b) the two types of parts, (c) the final construction.*

This method of making physical models of dual pairs can be applied to all Platonic solids. In Figure 4(a) and 4(b), we can see the model of the octahedron with the cube. In Figure 4(c), the complete set of the Platonic solids with their "extruded duals" are shown.



(a) (b) (c)

Figure 4: *(a) and (b) Parts and model of the dual-extrude model of octahedron and cube. (c) The five dual-extrude models of the Platonic solids, where I rounded off the extruded parts..*

In Escher's print "Kristal" the octahedron and the cube are both solid shapes. In the model made with the dual-extrude construction method, the cube is no longer solid, but clearly recognizable as a cube. Therefore, I have chosen to replace the cube with the extrusions of the edges towards the center. In Figure 4(c) we see all five possible combinations that can be made with the Platonic solids. In general, the assembly of the model is easy and the resulting construction does not fall apart quickly.

Dual Extrude – Archimedean Tilings

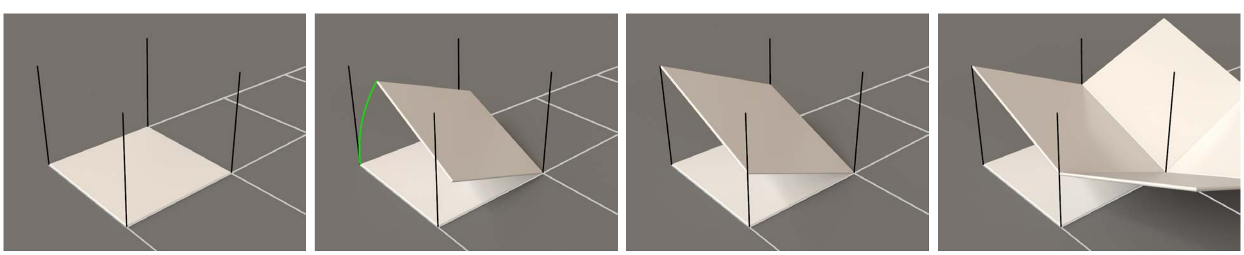
We can also apply the dual-extrude construction method to Archimedean tilings. In the Platonic solids, the extrusion direction was always towards the center of the solid. Here the extrusion is in both directions and we have to set a certain height for the extrusion of one of the tilings. The 333333-tiling and the 666-tiling are each other's dual tiling. When we want to combine these two tilings with the dual-extrude construction method, we have to choose which tiling to extrude. Both possibilities are shown in Figure 5(a). Another example is shown in Figure 5(b). Here, the basic Archimedean tiling is the 34334-tiling whose dual is the Cairo tiling.

(a) (b) (c)

Figure 5: *The dual-extrude construction method applied to the Archimedean tilings (a) 333333 and 666, (b) dual of 33434, 33434, and (c) 4444 respectively.*

Rotate-Stretch

A special case is the Archimedean 4444-tiling which is its own dual. So now there is only one possibility. It is also special in another way: when we take up this construction, it falls apart very easily. This is because we have sets of parallel sliding lines. We can solve this by transforming the square as follows: we start by rotating a square tile around one of the corners so that the diagonal opposite corner tilts as shown in Figure 6(b). Then we stretch the tiling so that, when we look from above, it covers exactly the original tile, Figure 6(c). To complete the full operation for the tiling, we mirror the tilted tile along the grid lines of the square grid, Figure 6(d).



(a) (b) (c) (d)

Figure 6: *Rotate-stretch of the square tilings of the Archimedean tiling 4444.*

Metamorphosis of the Archimedean Tiling 4444

There is no specific angle for the rotation and it turned out that we get a nice set of constructions when we used a slowly increasing angle in a few steps. In Figure 7 we see the original square tiling in the first image and the transforming of the pattern in four steps. The resulting constructions, except the first one, now are all stable.

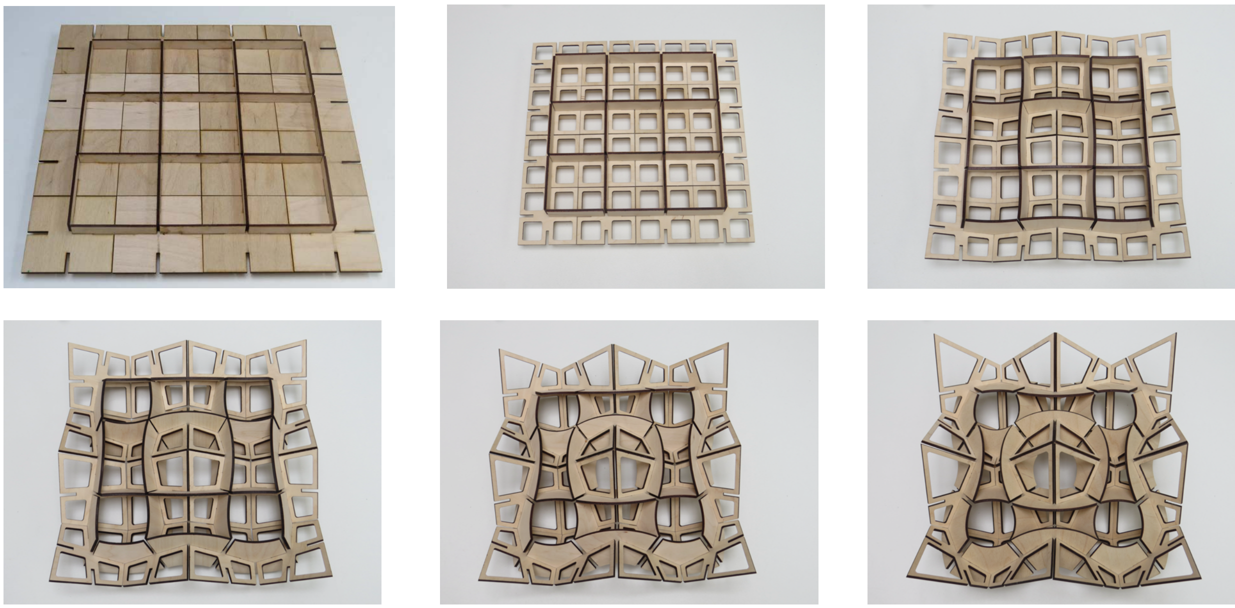


Figure 7: *Transforming the Archimedean tiling 4444 with the "rotate-stretch" method.*

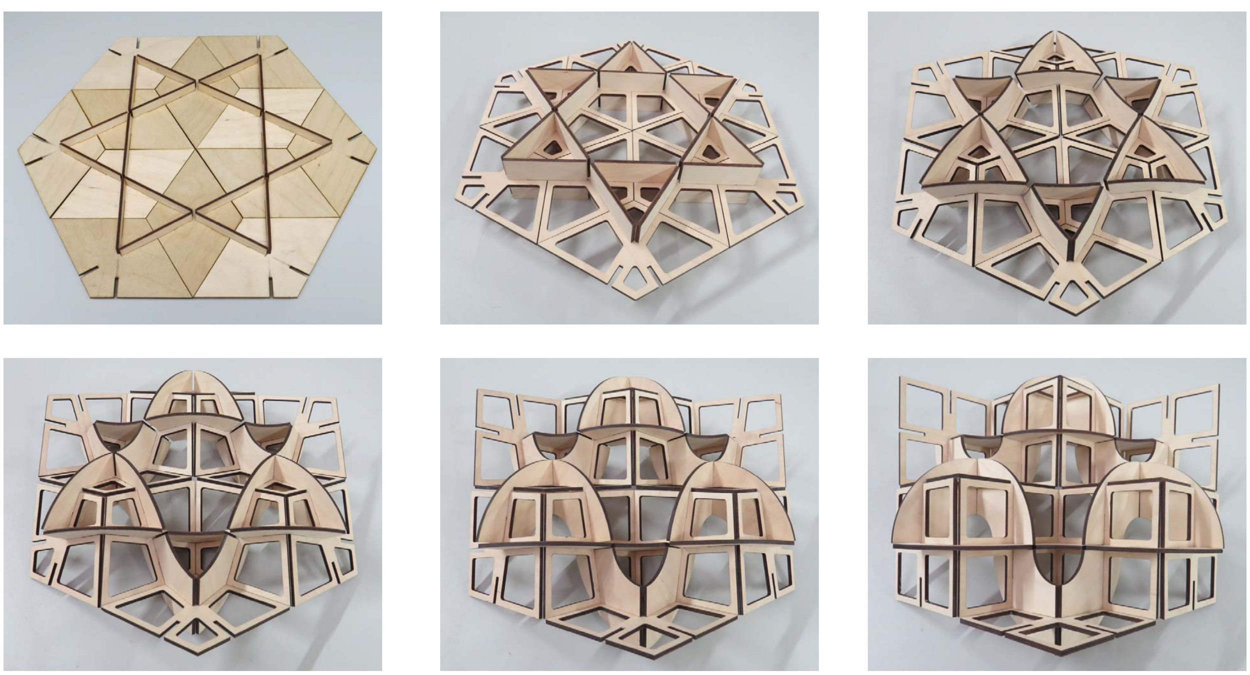


Figure 8: *Transforming the dual of the Archimedean tiling 3636 with the "rotate-stretch" method.*

The dual of the Archimedean tiling 3636 is a tiling with rhombuses. This tiling is also not stable; it falls apart too easily. The same “rotate-stretch” technique can now be used to create new, stable structures. The images in Figure 8 illustrate this process in four steps.

With the “rotate-stretch” technique the structures of all steps look the same when seen from above. It is just the shape of the tiles that changes. When we compare the two sets of metamorphoses as illustrated in Figure 7 and Figure 8, the one that starts with square tiles and the one that starts with the rhombs, we see that in the first set the shape of tiles transforms from squares to rhombs and in the second set the shape of the tiles changes from rhombs to squares.

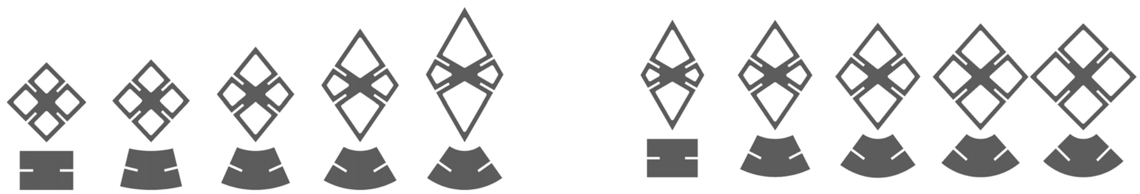
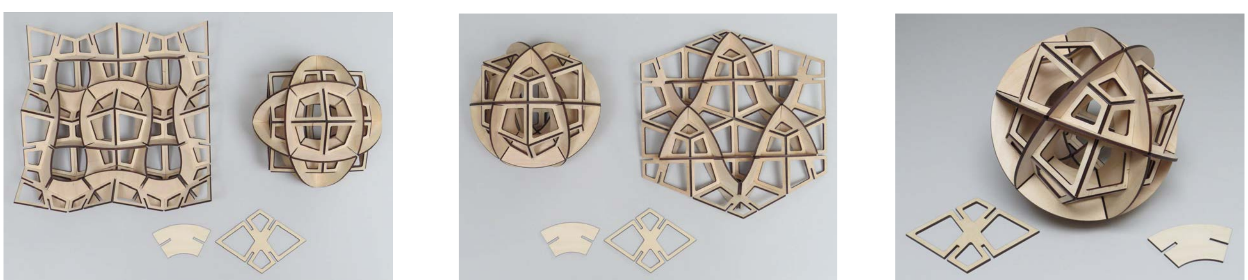


Figure 9: *Transformation of the shapes of the tile: from square to rhombs and from rhombs to squares.*

This gave me the idea that there might be a point in the transformation of the squares where the shape of the tiles is equal to the shape of tiles at a point in the transformation of the rhombs.

Rhombic Dodecahedron

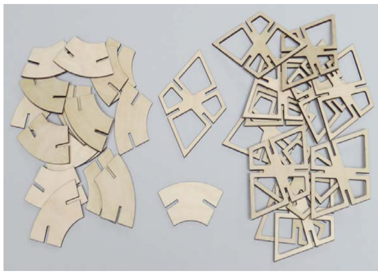
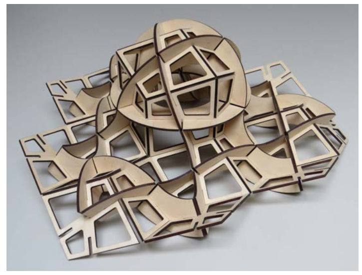
Such a “crossing point” does indeed exist. And the shape of the tiles turned out to be exactly the shape of the faces of a rhombic dodecahedron. We can understand this when we realize that a rhombic dodecahedron has fourfold symmetries, corresponding to a point in the transformation of the squares, and also a threefold symmetry corresponding to a point in the transformation of the rhombi. So now with exactly the same set of tiles, together with the dual-extrusion parts, we can make three different constructions (Figure 10).



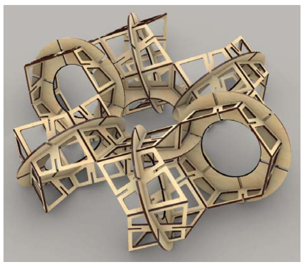
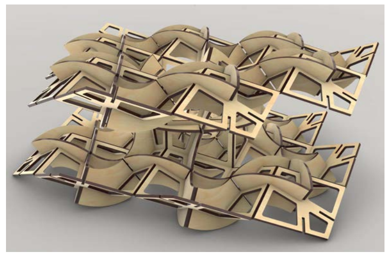
(a) (b) (c)

Figure 10: *(a) Rhombic dodecahedron compared with the transformation of the tiling with the squares and (b) compared with the transformation of the tiling with the rhombs. (c) Dual-extrude construction of the Rhombic Dodecahedron.*

But that’s not all. The rhombic dodecahedron is a space filler and that means that we can build a huge variety of combinations with the rhombic tiles and the dual-extrude parts. So, with a set of rhombic tiles together with a set of the dual-extrude parts as shown in Figure 11(a), we can now experiment and put together many different constructions. Figure 11(b) to Figure 11(f) show just a few possibilities.


(a) (b) (c)


(d) (e) (f)

Figure 11: *(a) The basic two parts for the example constructions in (b) – (f).*

Kites

There is one more tiling that can be processed with the rotate-stretch technique. The dual of the Archimedean tiling 3464 consists of kite-shape tiles and we can transform these tiles in the same way as the square tiles or the rhombus tiles in the previous metamorphosis in Figure 7 and Figure 8.

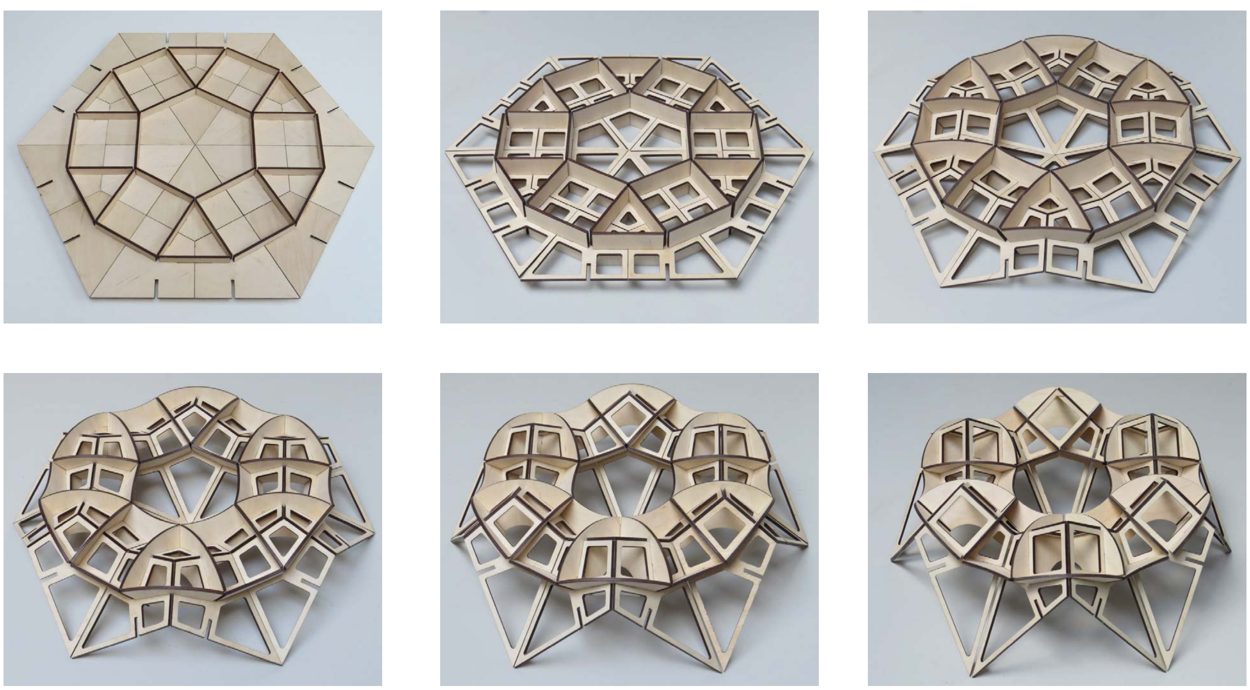


Figure 12: *Transforming the dual of the Archimedean tiling 3464 with the "rotate-stretch" method.*

The transformation is shown in Figure 12 and ends at the point where cubes seem to appear in the top ring of the construction. The four steps, together with the initial construction of the Archimedean tiling 3464 as a dual-extrusion construction, form again a nice metamorphosis. I have chosen the construction of the first step of the transformation for a bigger sculpture (Figure 13).

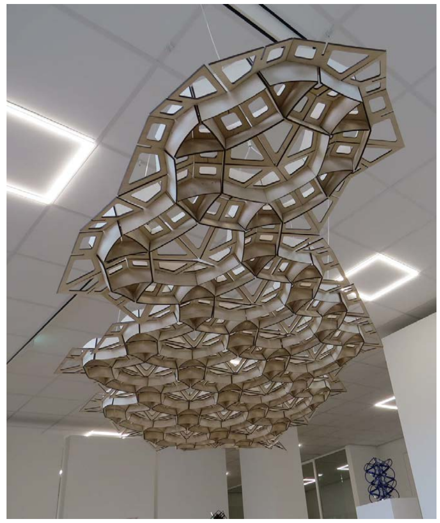


Figure 13: *Sculpture based on the first step of the transformation of the dual Archimedean tiling 3464 as shown in Figure 12.*

Sculptures

For me, the concept of the dual-extrude construction method have led to a new group of sculptures that are composed of two groups of parts: the faces of the basic geometric form and the extruded dual forms. The sculptures are normally easy to assemble by sliding the parts together and then form a stable structure.

(a)

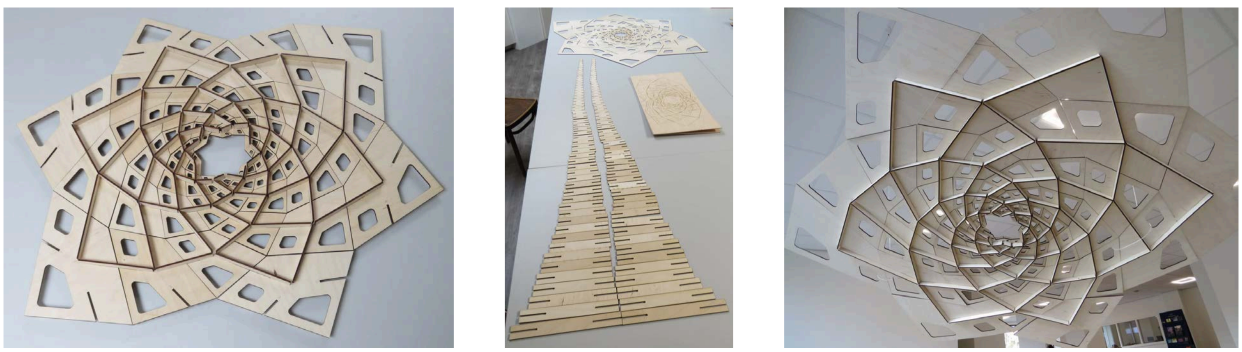
(b)

Figure 14: *Two columns, dual-extrude constructions, models for bigger sculptures.*

The model in Figure 14(b) was finally made of metal with a total height of 250 cm. Only two different parts were needed as can be seen in Figure 15(a). When we slide the parts together, we see small cubes appear along the axis of the construction.

(a) (b) (c)

Figure 15: *The two different parts and construction of the final sculpture 'Dancing Cubes'.*



(a) (b) (c)

Figure 16: *Dual-extrude-construction based on a logarithmic spiral tiling.*

A final example of such sculptures is based on the logarithmic spiral tiling. Here again there are two groups of elements: the basic tiles of the logarithmic spiral tiling and the extrusions of the dual. But here all elements have different dimensions as can be seen for the dual parts in Figure 16(b).

Conclusion

We have seen how the dual-extrude construction method can be used to create duality models based on the Platonic solids. This can easily be extended to the Archimedean solids. Then we have seen the general components for building structures based on the rhombic dodecahedron and the fact that this is a space filler. Finally, I have used this technique to design new sculptures. Here I took the liberty of varying the shape of the parts while maintaining the sliding structure. For me it opened up a new way of creating interesting structures and sculptures.

References

[1] W. Jamnitzer. *ThePerspectiva corporum regularium.* Edicione Siruela, 1993.

[2] J.L. Locher. *De werelden van M.C. Escher.* Meulenhoff, Amsterdam, 1971.