Reflected Motifs in Quasiperiodic Escher-Penrose Tilings

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Abstract

Quasiperiodicity is a slightly weaker form of periodicity and enables the design of tilings with an approximate structural fivefold rotational symmetry. Inspired by the periodic, plane-filling tilings of Maurits Cornelis Escher (1898─1972), one of us (Uli) created figurative interpretations of the quasiperiodic Penrose tilings, which were developed from 1973 onwards. As in Escher's work, the Penrose tiles are artistically shaped and joined together. Due to a special design of the edge structure, the mostly figurative tiles also appear as their mirror images.

Introduction

Maurits Cornelis Escher [2] was born in Leuwarden in the Netherlands in 1898 and is considered to be one of the most remarkable graphic artists in Europe. In this paper we refer to that part of his work in which he stitches figures together so that they completely fill the plane. Sometimes he did this in an irregular manner, but mostly on the basis of various periodic grids.

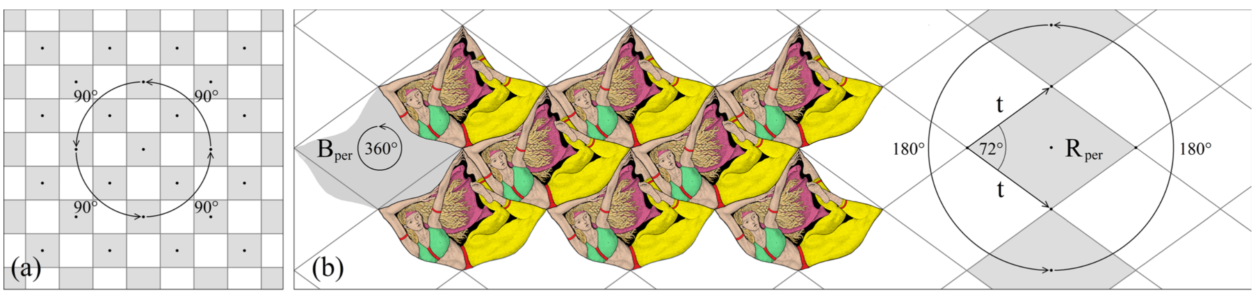


Figure 1: *(a) Fourfold rotational symmetry of a chessboard grid. (b) Sheared rhombic grid with dancers.*

The *fourfold rotational symmetry* of the chessboard grid in Figure 1(a), which should be considered as infinitely extended, is a *structural symmetry*. This means that the grid can be rotated by 90 degrees around the center of any squared grid mesh without changing the original grid. The chessboard grid represents the *symmetry group p4m*, one of the *17 crystallographic symmetry groups of the plane*.

The sheared rhombic grid in Figure 1(b) on the right has a twofold rotational symmetry. The shapes of the inserted dancers are *periodic boat tiles* (B_{per}), named after a similar paper boat. Although the tiles B_{per} have the same corner points as the *rhombs* R_{per} of the sheared grid, the twofold rotational symmetry of the grid is lost. Nevertheless, the *periodic translational symmetries t* of the (white) grid are retained.

It is proven that a periodic crystallographic order with a fivefold rotational symmetry cannot exist. However, from 1973, the British mathematician and physicist Roger Penrose developed *quasiperiodic tilings* with an *approximate structural fivefold rotational symmetry* [9]. The *Penrose rhombus tiling* from 1976, consists of *thick rhombs R* with acute angles of 72 degrees and *skinny rhombs* R_s with acute angles of 36 degrees. Different rules ensure that the ten geometrically possible orientations of the rhombs are equally frequent. Although the thick Penrose rhombs R have the same shape as the periodic rhombs R_{per} in Figure 1(b), the *matching rules* of the Penrose rhombs prohibit such periodic sequences.

In the chronological Q&A-list in the supplement you will find answers to term-specific questions.

Early Contributions to the Development of the Penrose Tilings

Along with Roger Penrose, Robert Ammann [10] is rightly cited as co-discoverer of the rhombus tiling. His drawing shown in Figure 2(a) dates from 1976 and is the earliest visualisation of a rhombus tiling. One year later, he developed the quasiperiodic *Ammann bars*, which can be considered as a fundamental decagonal q*uasiperiodic grid*. N.G. de Bruijn [1] of the Eindhoven University of Technology found some of the most important mathematical principles of the Penrose tiling theory. The sensational discovery of the *quasicrystals* by Daniel Shechtman in 1984 [11] made the Penrose tilings widely known, because they offer a way of modeling the quasiperiodic nuclear structure of *decagonal quasicrystals*.

The Global Construction of a Rhombus Tiling with the Quasiperiodic Ammann Grid

The original way of Penrose to globally create a Penrose tiling is the *substitution method* [3][9]. There each tile is substituted by a specified arrangement of smaller copies of the original tiles. We here present another way to create large tilings more quickly. In Figure 2(b) far left, three horizontal black lines are drawn on top of each other. The ratio of the two distances L_q and S_q is determined by the golden ratio τ, with $\tau = (1 + \sqrt{5})/2$. Then L_q is substituted by the interval sequence LSLSL and S_q by the sequence LSL, with $L/S = L_q/S_q = \tau$ and $L_q/L = S_q/S = \tau^3$ [3][7]. The continued, iterated substitution of this L-S interval sequence develops quasiperiodically (see the supplement) and is commonly called a *Fibonacci chain*.

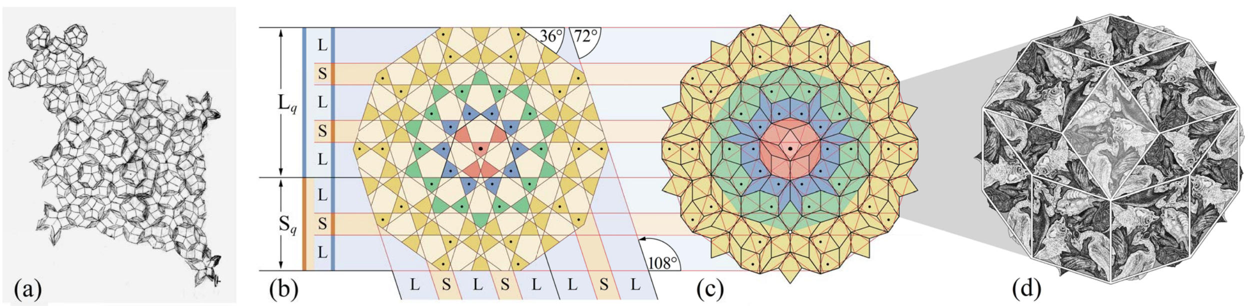


Figure 2: *(a) Drawing of a rhombus arrangement by Robert Ammann from 1976. (b) Construction of an Ammann grid. (c) Insertion of rhombs into the Ammann grid. (d) Animal decoration of Penrose rhombs.*

The nine horizontal boundary lines of the eight intervals L and S are now elongated to the right side of Figure 2(b). The nine lines give a quasiperiodic one-dimensional (1D) grid. This grid is four times rotated around the thick black point which has the same distance to the top and the bottom grid line. The rotation angles are 108 and 144 degrees, both clockwise and counterclockwise, so that the same *asymmetrical pentagonal grid meshes* are created in all regions of the grid, especially in the center! Please compare the counterclockwise 108-degree rotation of the 1D grid at the bottom edge. The result is called a *cartwheel-type grid*. In some of the colored grid meshes, the mesh centers are marked with small black dots to make the positions of the spokes of the name-giving cartwheel apparent. The dots also help when comparing Figure 2(b) with Figure 2(c), which shows the equivalent *rhombus cartwheel-type tiling*. If you zoom into the tiling, you can see that each of the irregular pentagonal grid meshes is surrounded by a thick rhombus whose long diagonal lies on the symmetry axis of the grid mesh. The size and the position of the thick rhombs is chosen so that the four rhombus edges each contain one corner of the corresponding grid mesh. The gaps between the thick rhombs can now be filled with skinny rhombs in a predefined way.

Mirrored Decorations of the Penrose Rhombs

In Figure 2(d), the small red decagon in the center of Figure 2(c) is enlarged by the factor τ^3. It is filled with animals with the same corner point structure as the rhombs. A closer analysis shows that the animals of the two lower thick rhombs are mirrored in relation to the central rhombus. In contrast, the two upper thick rhombs, which are rotated by 72 degrees to the central one, are not mirrored. We can therefore assume that two rhombs which are rotated 36 degrees to one another are always mirror images and that the skinny rhombs play a role in this. In the following sections we will show that this is the case.

The Construction of the Penrose Rhombus Tiling with Locally Acting Matching Rules

An alternative to the global plan design is the construction of a tiling using local matching rules. These rules are scientifically interesting because they correspond better to a quasicrystalline nuclear growth than the global methods, but earlier or later they lead to a dead end. The matching rules are also of interest to mathematical artists, as the required edge marks can be artistically shaped.

The basis of the local rules are the *neighborhood transformations h*. They are allowed by the same edge marks that prohibit periodic rhombus constellations. Usual edge marks are *De Bruijn arrows* or the *Ammann line segment*s which are shown in Figure 3(a). But these edge marks are not suitable to describe the mirroring of rhombs, since they themselves are mirror-symmetric. Therefore, we use the *asymmetric Ammann notches* to illustrate the mirroring transformations h_2* (Figure 3(b)) and h_4*.

The Five Transformations h_1, h_2, h_3, h_4, h_5 and their Mirroring Versions h_2* and h_4*

Each transformation in Figure 3(a) is defined as a rotation of a rhombus R_{id} which has a vertical oriented position, called *the identity* (*id*). The neighbors of a rhombus R_{id} in a reflected Escher-Penrose tiling are defined by the five *transformations* h_j, with $j \in \{1, 2^*, 3, 4^*, 5\}$ and their inverses h_j^{-1}, with a reversed direction of rotation. The pivot points W_2 and W_4 for the transformations h_2* and h_4* are constructed by elongating the upper edges of R_{id} diagonally downwards by the length τ, so that $|\overline{TW}| = 1 + \tau$.

The symmetry-breaking Ammann notches in Figure 3(b) make it possible to distinguish the rhombs R and R_s from their *mirror images* R* and R_s*. The triangle U_1 within R_{id} is mirrored by a *glide reflection* g into a triangle U_1* and then shifted along the *glide path* into the lower half of the skinny rhombus R_s*. Now only a mirrored rhombus R* has the appropriate counterpart to the reflected Ammann notch of U_1*! The upper half of the rhombus R_s*, the triangle U_2*, is a reflection of U_2 at the horizontal dashed line m. Figure 3(c) shows the *transformation* h_2*(R_{id}), visualized by the animals of Figure 2(d).

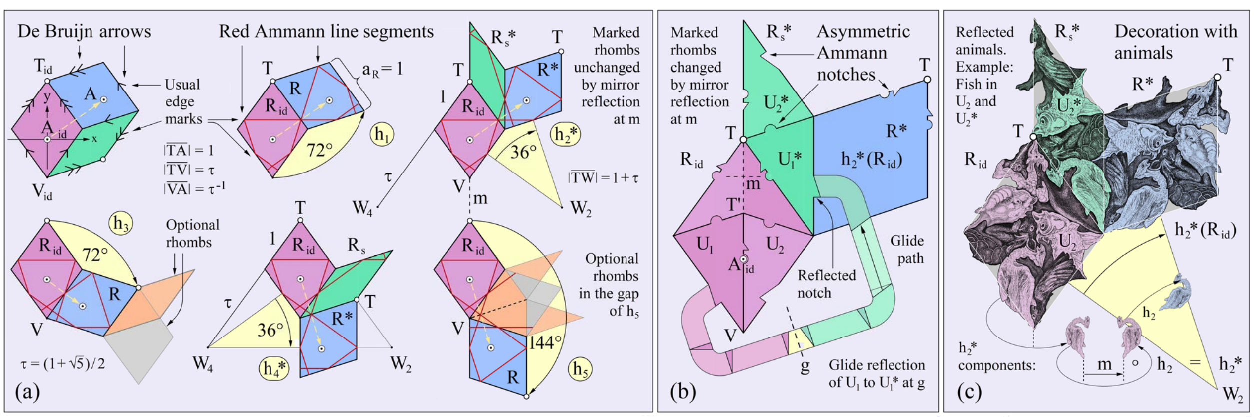


Figure 3: *(a) Transformations h_j. (b) Reflections of Ammann notches. (c) Decorated transformation h_2*.*

- h_1 stands for a counterclockwise 72 degree rotation of the thick rhombus R_{id} around its upper point T.
- h_2 stands for a clockwise 36 degree rotation around the point W_2 (in this paper h_2 is substituted by h_2*).
- h_3 stands for a clockwise 72 degree rotation of the thick rhombus R_{id} around its lower vertex point V.
- h_4 stands for a clockwise 36 degree rotation around the point W_4 (in this paper h_4 is substituted by h_4*).
- h_5 stands for a clockwise 144 degree rotation of the thick rhombus R_{id} around its lower vertex point V.
- h_2* is equal to h_2 with a preceding reflection of R_{id} about its long diagonal (h_2* is used in all examples).
- h_4* is equal to h_4 with a preceding reflection of R_{id} about its long diagonal (h_4* is used in all examples).

Alternatively, the transformations h_j can be described *arithmetically* with the *complex numbers z* in the *complex plane* $\mathbb{C}$. There, each transformation h_j consists of a rotation of R_{id} around the point A_{id}, followed by a *shift* of the distance l or s (only h_3). In Figure 3(a), the shifts are indicated by *dashed yellow arrows*.

Specified information on the complex equations can be found in [8] and in the supplement.

The Transformations h_2*, h_4* and h_3 in the Penrose Kite & Dart Tiling

The Penrose kite & dart tiling is closely related to the Penrose rhombus tiling. The order of the animals in Figure 4(a) is exactly the same as in the rhombus tiling in Figure 2(d). The color scheme helps us to see the equivalence relation between the animals and the kite & dart tiling in Figure 4(b). There, the points A_{id} of the thick rhombs R are marked in white, i.e. a thick rhombus R always consists of one *D-tile* and two adjacent halves of two different *K-tiles* (Figure 4(b), top left). The idea to compose a thick rhombus from one kite and one dart contradicts the Ammann notches, as shown by the two crossed-out arrows.

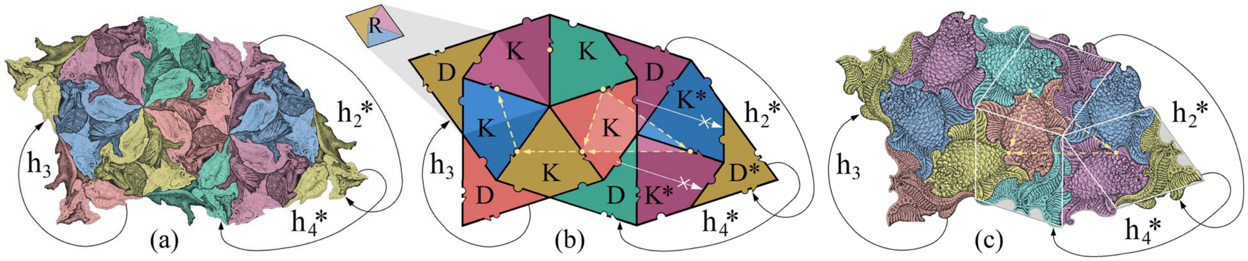


Figure 4: *A successive execution of h_2* and h_4* gives the transformation h_3. (a) Kite and dart coloring of the rhombus animal tiling. (b) Kite and dart tiling. (c) Kite fish and dart rays with superimposed rhombs.*

The dashed legs of the triangle in the center of Figure 4(b) represent the shifts l of the points A_{id} of the transformations h_2* and h_4*. The shorter basis of the triangle represents the shift s of h_3. In the pattern in Figure 4(c), called *kite fish and dart rays*, the superimposed rhombs show that the reflections of the transformations h_2* and h_4* cancel each other out and that these transformations, executed in succession, correspond to a transformation h_3, which transforms the purple dart ray into the turquoise one.

The One-Dimensional Quasiperiodicity Shown by the Kites *K and K** in a Cartwheel

Figure 5(a) shows the kite fish and the dart rays in a cartwheel order. We use the equivalent geometric kite & dart cartwheel in Figure 5(b) to visualize the quasiperiodic order by breaking them down into five one-dimensional (1D) components.

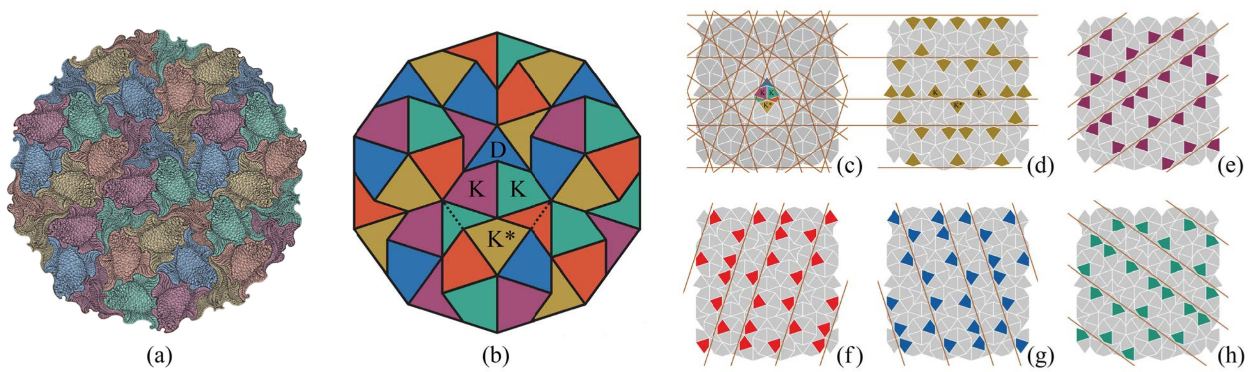


Figure 5: *(a) Cartwheel made of kite fish and dart rays. (b) Cartwheel made of kites (K) and darts (D). (c) Ammann bars related to the central rhombus. (d-h) One-dimensional quasiperiodic kite sequences.*

The Ammann bar grid in Figure 5(c) corresponds to the central rhombus, which can be seen enlarged in the center of Figure 5(b). This rhombus is composed of two K, one K*, one D and two halved D-tiles. Figure 5(d) shows, that the yellow K and K* tiles are lined up above and below the horizontal lines. The sequences along the lines are called quasiperiodic because no short or long tile sequence is repeated more than once, i.e. there is no translational symmetry! The five 1D-grids in Figure 5(d-h) are not significantly different from one another, not even in a very large cartwheel. Consequently, the fivefold symmetries are evenly but quasiperiodically distributed, even in a very large cartwheel structure.

Color Coded Matching Rules for the Girih Patterned Hexagon Boat (*HB*) Tiling

Marks on the tile edges are often artistically undesirable and their artistic deformation, as already shown in the animal motifs, is sometimes also not appropriate. As an example, we show here a tiling with Girih ornaments, in which the elongated hexagons *H* and the boats *B* (see Figure 1(b)) with straight edge lines are required. This *Penrose hexagon boat (HB) tiling* was first presented by one of us (Uli) in 2015 in the gallery "*Quasicrystalline Wickerwork*" [5]. It was specially developed for physical puzzle pieces, because neither of the two proto-tiles has a fragile acute angle of 36 degrees.

The Penrose *HB* tiling is derived from the Penrose rhombus tiling in that each rhombus edge is substituted by two shorter edges that are at an angle of 144 degrees to each other. This deforms the thick rhombus *R* into a boat *B* and the skinny rhombus R_s into an elongated hexagon *H*. During this deformation the skinny rhombus R_s loses one of its corners as two of the short edges coincide. This is shown in Figure 6(a) top right as a double line filled in green. The green filled double line thus becomes a common edge of a boat *B* and a boat *B**, which is turned on its back (please note the difference from mirroring).

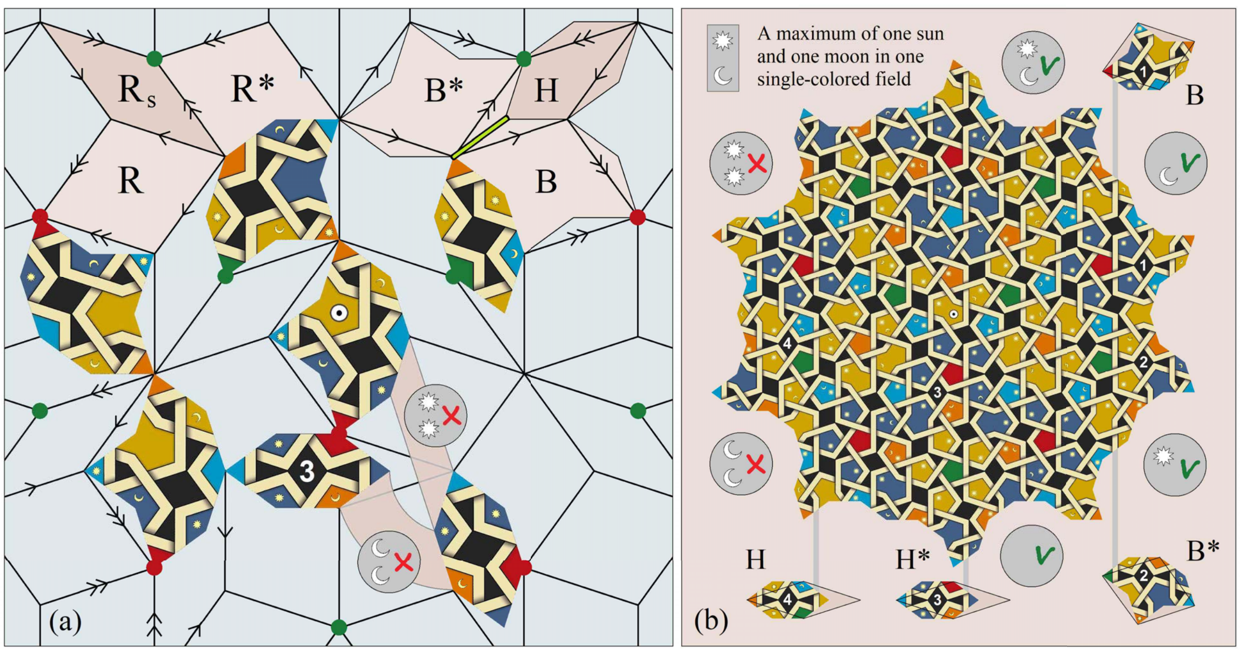


Figure 6: *(a) Correspondance of the Girih patterned HB-tiling to the De Bruijn arrowed rhombus tiling. (b) Girih cartwheel with the necessary additional sun-moon rule that completes the matching rules.*

From a purely formal point of view, the edges of the tiles *H*, *H**, *B* and *B** are all identical and the Girih pattern could be consistently continued across the tile edges in every tile constellation, i.e. with uncolored Girih tiles, a periodic but geometric *B*-tile arrangement as shown in Figure 1(b) would be possible.

However, since the geometric shapes of the tiles are neccessary in order to fit the Girih pattern into them, the matching rules here are alternatively given by a color scheme, i.e. only the same colors may come together on one edge. The colors of the two inverted tiles *H** and *B** are given the complementary colors of the *H* and *B* tiles, i.e. red becomes green, blue becomes orange and yellow becomes violet and vice versa. Therefore, the Girih strands always enclose uniformly colored fields.

The final fulfillment of the matching rules is achieved by the *sun-moon rule*, which is illustrated in the six circular pictograms in Figure 6(b). The four pictograms on the right show that the sun and the moon may be together in one color field. Also allowed is only the moon or only the sun or none of both. The two pictograms on the left show that two moons or two suns in the same color field are forbidden. Figure 6(a) shows which tile placements are prohibited by these two pictograms.

The Quasiperiodic Cartwheel Ballet Created on the Basis of the *HB*-Tiling

The Correspondance of the Female and the Male Dancers to the HB-Tiles

The corner point structure of the ballet dancers in Figure 7(a) corresponds to the *HB*-tiling. but unlike the Girih-patterned version, the edges of the ballet dancers are slightly deformed. However, the deformation of the female *B* dancer shown in Figure 7(a) differs from that one in Figure 1(b), which forces periodicity. In contrast, the differently curved edges of the dancers in Figure 7(a) enforce quasiperiodicity. Figure 7(b) illustrates the equivalence relationship between the *HB*-tiles and the rhombus tiles.

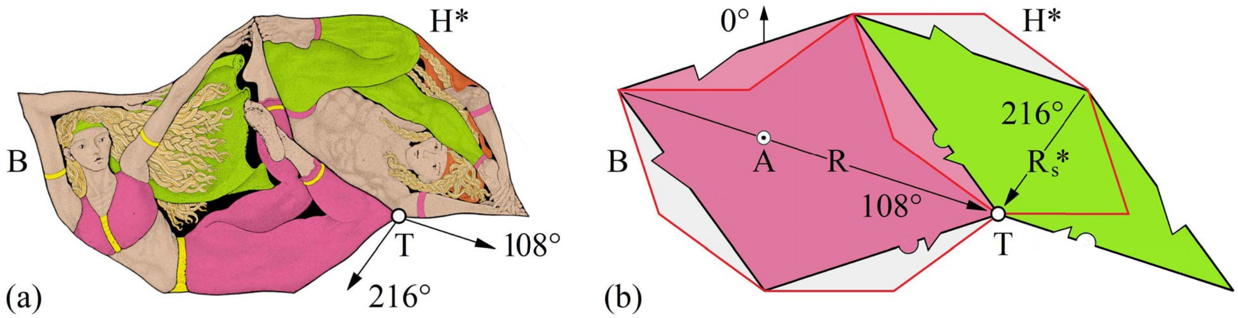


Figure 7: *(a) B and H* dancers side by side. (b) B and H* tiles superimposed on the Penrose rhombs.*

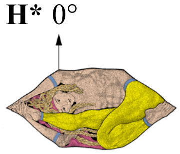
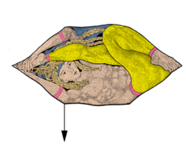
In the supplement you will find an illustration that shows that the female *B* dancers in figure 7(a), which represent quasiperiodic matching rules, cannot be used to compose a periodic order shown in Figure 1(b).

The Color Scheme of the Ballet Dancers

A total of 20 color combinations is used, firstly to make the 1D quasiperiodicities of identically oriented and same-colored dancers separately visible (see also Figure 5), secondly to characterize the centers of rotation with different colors, and thirdly to cluster garments of the same color in suitable regions.

In Table 1, the orientations of the tiles are indicated by an angle given by the vector $\overline{AT}$ (Figure 7(b)) with its basic position pointed vertically upwards (0°). The direction of rotation is clockwise. The stripes on the hems and the bracelets have a different color than the clothing, as well as the hair veils and hair ribbons. The angles in the purple and green boxes indicate the orientations of the dancers in Figure 7(a).

Table 1: *The coloring of clothes, stripes and hair veils of the differently oriented B and H dancers.*

B	***Angles***	***Clothes***	***Stripes***	***Veils***	**B*** **T** 0°	***Angles***	***Clothes***	***Stripes***	***Veils***
	180°	yellow	blue	purple		0°	yellow	purple	blue
	252°	blue	orange	yellow		72°	blue	yellow	orange
	324°	orange	green	blue		144°	orange	blue	green
	36°	green	purple	orange		216°	green	orange	purple
T 180°	108°	purple	yellow	green		288°	purple	green	yellow
H* 0°	0°	yellow	blue	purple	**H**	180°	yellow	purple	blue
	72°	blue	orange	yellow		252°	blue	yellow	orange
	144°	orange	green	blue		324°	orange	blue	green
	216°	green	purple	orange		36°	green	orange	purple
	288°	purple	yellow	green	180°	108°	purple	green	yellow

The Quasiperiodic Cartwheel Order of 35 Female and 15 Male Dancers

The construction of a cartwheel has already been described in Figure 2. The Girih patterned *HB* cartwheel in Figure 6 has the same size and the same number of tiles as the *HB* cartwheel with the ballet dancers in Figure 8, apart from the fact that the dancers have slightly curved edges and that the two cartwheels are laterally reversed to each other.

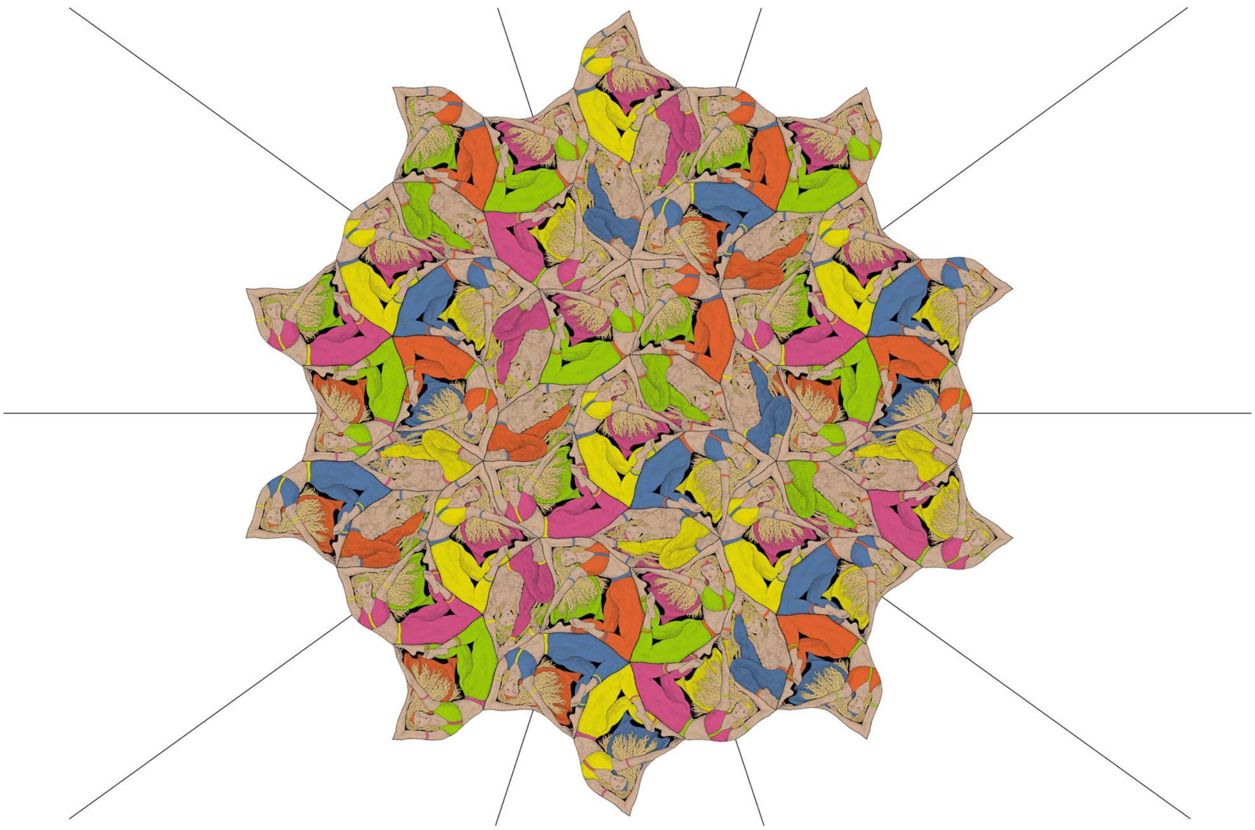


Figure 8: *The cartwheel ballet*

In Figure 8, the different distances between the equally oriented dancers, best seen at the yellow-clad female dancers with the purple veils, give an idea of the one-dimensional quasiperiodicity. Only in very large cartwheels it becomes visible that no sequence of this distances is repeated periodically. However, the cartwheel ballet also shows many other interesting symmetrical features. While the outline of the corresponding rhombus cartwheel in Figure 2(c) has a tenfold rotational symmetry, the *HB* cartwheel has an outline with fivefold symmetry because the female dancers, which form the ten prongs of the outline, are alternately *B* and *B** dancers. Nevertheless, each of the ten outer female dancers has five mirror images, each of which is a reflection on one of the five drawn axes. The outer mirrored dancers standing opposite each other have the same color of clothing, but differently colored veils and stripes. Despite the asymmetrical order in the center, a few of the inner dancers are also reflected on some of the axes.

A large version of the Cartwheel Ballet will be presented in the 2025 bridges exhibition [6].

Summary and Conclusion

Using a chessboard as an example, we described what is meant by a periodic (crystallographic) structural fourfold rotational symmetry. On the basis of a periodic sheared rhombic structure, we illustrated how the plane can be completely filled with figures that have the same area and the same corner point structure as the rhombs. Figurative periodic tilings of that kind are today commonly called Escher tilings.

We have shown that the translational symmetries are a characteristic feature of all periodic plane fillings, in contrast to the quasiperiodic Penrose tilings that we have used as the basis for some figurative plane fillings with an approximate fivefold rotational symmetry.

We presented the concept of mirrored motifs, as well as the previously unknown Penrose *HB* tiling, which can only be realized with mirrored tiles. Using the example of an *HB* tiling decorated with Girih ornaments, we showed that the quasiperiodic matching rules can be controlled by implementing suitable color fields. In addition, we introduced a sun-moon rule that acts as both an extension and a complement to the color coded matching rules. For the cartwheel ballet, we demonstrated that the matching rules of a Penrose *HB* tiling can also be controlled by slightly curved edges, which on the one hand act as edge marks and on the other hand enable a more sculptural design of the dancers.

Outlook

There are several reasons why we consider the Penrose *HB* tiling to be particularly suitable for figurative applications. First, the tiling consists of only two proto-tiles, which limits the coordination effort for the edge modification. Second, the proto-tiles are compact because they do not have acute 36-degree angles. Third, the reflections can be used to create artistic stimuli, e.g. through colors, gestures or a complex interplay of the figures' eye contact. For this reason, we are planning a paper in the near future that will mainly deal with the *HB*-tiling and its variants, including their matching and substitution rules.

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