Holomorphic Mappings for Integrated Garment and Motif Design

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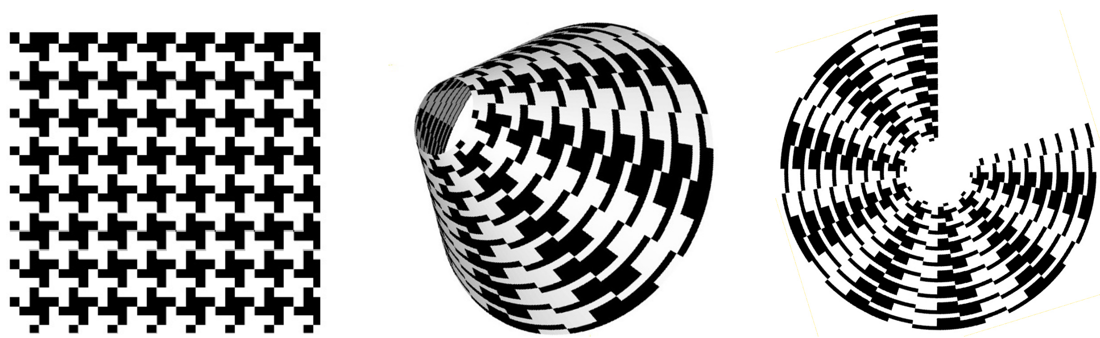
Abstract

Today's fashion design is based on a separation of concerns such that the geometry of the fabric motif and the geometry of the pattern cut are unrelated. We challenge this approach by developing mathematical tools to morph the motif. The core innovation lies in the mathematical framework, based on holomorphic mappings and harmonic conjugate functions, to map motifs onto arbitrarily shaped panels. This approach, implemented via custom software, allows for seamless motif continuation across complex garment shapes, avoiding cutting through repeating designs. We demonstrate the technique's application through several garment examples, showcasing its potential for creative pattern design and efficient manufacturing.

Introduction

It is an unwritten rule of fashion that it is permissible to cut through the motif whenever the overall design of the item requires so. Viewed from a historical perspective, when fabrics were constructed on traditional looms and the motifs had to emerge from the weave as well, this rule is well-justified. However, nowadays we have printers and computerized Jacquard looms and it is time to reconsider the unwritten rule and explore alternatives. We may face new challenges, but we may also discover new aesthetic possibilities.



(a) (b) (c)

Figure 1: *Morphing: (a) puppytooth motif, (b) mapped on a cone, (c) using a pattern.*

We use the term "pattern" for an assembly of sewing lines, necklines, hemlines, darts etc. The pattern is what concerns the seamster. We use the term "wallpaper" in the mathematical sense, viz. an assembly of images that are repeated by two distinct translations, possibly equipped with additional symmetries [4]. The "motif" is the element that is repeated, for example a bunch of flowers, a brand logo (such as the two letters F of Fendi), bird or a fish (in case of an Escher-style design), or a pied-de-poule tile (houndstooth). The wallpaper and the motif are what concern the graphic designer and the printer. A typical fashion item, such as a garment or a bag is composed of a number of "panels", each of which is described by a pattern.

As our first example, consider a cone, which is the simplest pattern of a skirt imaginable. If we cut the layout of a cone from a wallpaper, we must cut through the motif. We can do better by mapping the grid of the wallpaper to the layout pattern of the cone, as illustrated in Figure 1.

How do we obtain such a pattern? In a Cartesian coordinate system where we know the color of the motif for each pair (x, y), then we need a mapping $(x, y) \mapsto (u, v)$. The puppytooth motif is naturally described in x, y coordinates. In this case, the unit cell of the motif is square and without loss of generality we choose the width and height of the unit cell to be 1. To begin, we introduce polar coordinates r and φ such that $u = r \cos \varphi$ and $v = r \sin \varphi$. For Figure 1, we let r increase linearly with x and we choose φ to be proportional to y, say $\varphi = (2\pi/N)y$ which allows us to fit N unit cells along one circle, e.g. $N = 8$ in Figure 1(a). If it is a 280° segment as in Figure 1(b,c), instead of a full circle, we set $\varphi = \frac{280}{360}\left(\frac{2\pi}{N}\right)y$. A piece of wallpaper with $M \times N$ motifs is an area $\{(x, y) \mid 0 \leq x \leq M, 0 \leq y \leq N\}$ for positive integers M and N. If we do not use any mapping, we can sew the line $\{(x, 0) \mid 0 \leq x \leq M\}$ to the line $\{(x, N) \mid 0 \leq x \leq M\}$ and thus obtain a cylinder for which the motif goes around without disruption.

If we sew the corresponding line *after mapping*, we obtain the cone of Figure 1(b). Upon closer inspection, however, we notice that something is not right, the puppytooth tiles are stretched more and more as we go down the cone. However, they are stretched in one direction and not in the other. To remedy this we need to make the steps Δr between mapped unit cells such that the step becomes larger as r becomes larger[1]. That sounds familiar: a function whose derivative behaves like the function itself. Indeed, we need an exponential function $r = e^{\omega x}$ (for some constant ω). We will return to this formula later.

Beyond Cones: Complex Numbers and Holomorphic Mappings

Aiming at more interesting garments, we need a more general class of mappings. We focus on a rich class of morphing mappings, known as holomorphic. The theory of complex functions of complex variables [1] provides valuable concepts to define such mappings. We denote complex numbers by $z = x + iy$ where x and y are real numbers and i has the special property that $i^2 = -1$. In Figure 2 we illustrate how a simple function on complex variables works as a mapping.

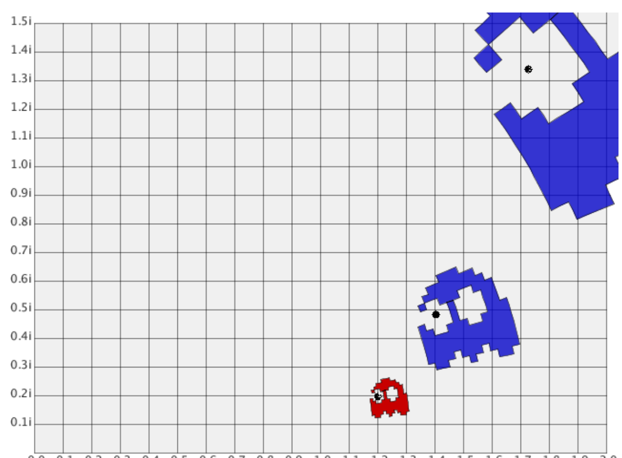


Figure 2: *Morphing the Pacman ghost motif by the function $f(z) = z^2$.*

In Figure 2 we use the function $f(z) = z^2$. We show the area of numbers $x + iy$ for $0 \leq x \leq 2$ and $0 \leq y \leq 1.5$ where the center of the left eye of the red Pacman ghost represents the number $z = 1.2 + 0.2i$. Let us calculate the square of that number: we find $z^2 = (1.2 + 0.2i)^2 = 1.44 + 2 \times 0.24i + 0.2^2 \times i^2 = 1.44 + 0.48i + 0.04 \times (-1) = 1.4 + 0.48i$. This z^2 is the center of the blue ghost's left eye. In the same way, all other points of the red ghost are mapped by the same function, which is how we found the blue

[1] This may sound like Riemann integration in polar coordinates. However, there the goal is to keep Δr constant and adjust the area formula. Here we let Δr increase as we move outward.

ghost. We could continue by calculating z^4 and thus see how the next ghost explodes beyond our number range (this example is taken from the artwork presented in my Bridges 2021 paper [2]).

The concepts of differentiation and integration, which are well-known for functions on the real numbers, can be generalized to functions which are defined on the complex numbers, and which return complex results. The full theory is in [1], here we can only mention a few key elements. A holomorphic function is a function $f(x+iy) = u + iv$ that is complex differentiable on some domain, where complex differentiability is given by the two conditions $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\,\partial v/\partial x$, known as the Cauchy-Riemann equations[2]. In practice, the terms *holomorphic* and *analytic*[3] are used interchangeably. Holomorphic functions with non-zero derivatives are *conformal*: they morph a 2D area into another area while preserving angles. If we consider very small squares (squares of infinitesimal size) then these are mapped to squares again (Figures 2 and 3).

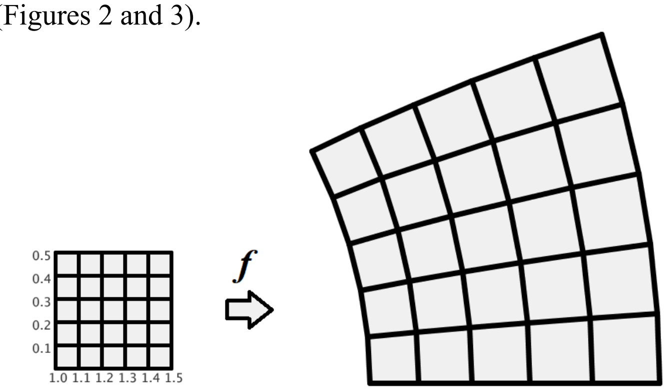


Figure 3: *Conformal mapping of a coordinate grid.*

This theory is very powerful, so let us apply it to our earlier cone design – using exponentiation instead of squaring. We are allowed to have the special number i inside the exponent using Euler's formula $e^{i\varphi} = \cos\varphi + i\sin\varphi$. If in polar coordinates we have $\varphi = (2\pi/N)y$ and $r = e^{\omega x}$, we can rewrite the mapped coordinate pair (u,v) as a complex number $w = u + iv$ and then the mapping $(x,y) \mapsto w$ has a single equation $w = e^{\omega x} \times e^{i(2\pi/N)y}$. This is because multiplying with $e^{i\varphi}$ represents a rotation over φ (radians). Introducing $z = x + iy$ and choosing $\omega = 2\pi/N$ we get simply $w = e^{\omega z}$. The choice for ω is not just based on the elegance of the formula, it is precisely the condition that guarantees that the motif is equally stretched in the r and φ directions. This is the analytic way to find a good, i.e. conformal, mapping for disks and cones (it works because *any* function of the form $f(z) = e^{\omega z}$ is holomorphic).

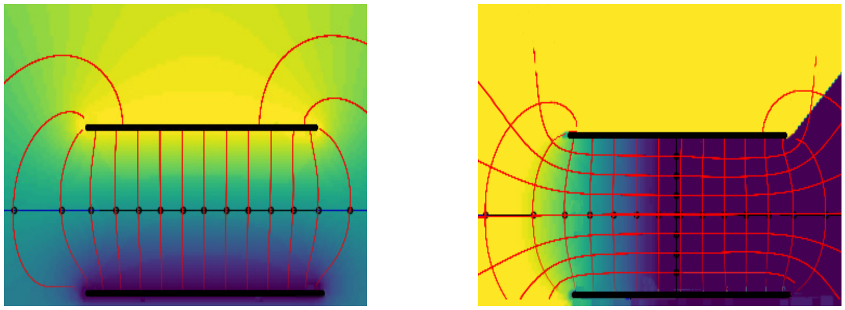
If the contour of the morphed grid in Figure 3 would be the contour of our garment panel, then we could draw our motifs in the left grid, and next map by f. It would even be sufficient to have only the morphed grid, as we can then redraw the motifs while interpolating inside the grid cells. The question is: how do we find a grid inside a contour defined by arbitrary lines? Surprise: by electrical engineering.

It is a hidden gem in electrical engineering that the theory of 2D electrostatic fields coincides with the theory of analytic functions. A real-valued function $\psi(z)$ where $z = x + iy$ satisfying $d^2\psi/dx^2 + d^2\psi/dy^2 = 0$ is said to be *harmonic*. Writing this condition as $\nabla^2\psi = 0$, known as Laplace's Equation, one recognizes that any harmonic ψ is the potential of a charge-free electric field. The real and imaginary parts of a complex analytic function both satisfy the Laplace equation. Conversely, if we have a harmonic function $\psi(z)$, then we know that it is the real part of a complex function $\zeta(z) = \psi(z) + i\chi(z)$ for some function $\chi(z)$. The χ is essentially defined once we know ψ and it is called the *harmonic conjugate* of ψ ([1], p.201). Richard Feynman in Chapter 7 of [3] writes about this: "Now we come to a miraculous

[2] It can be checked that our function $f(z) = z^2$ satisfies these conditions. We have $u = x^2 - y^2$ and $v = 2xy$ so $\partial u/\partial x = \partial(x^2-y^2)/\partial x = 2x = \partial(2xy)/\partial y = \partial v/\partial y$ and $\partial u/\partial y = \partial(x^2-y^2)/\partial y = -2y = -\partial v/\partial x.$

[3] A function $f(x+iy) = u + iv$ is analytic if it has a convergent power series, e.g. $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$.

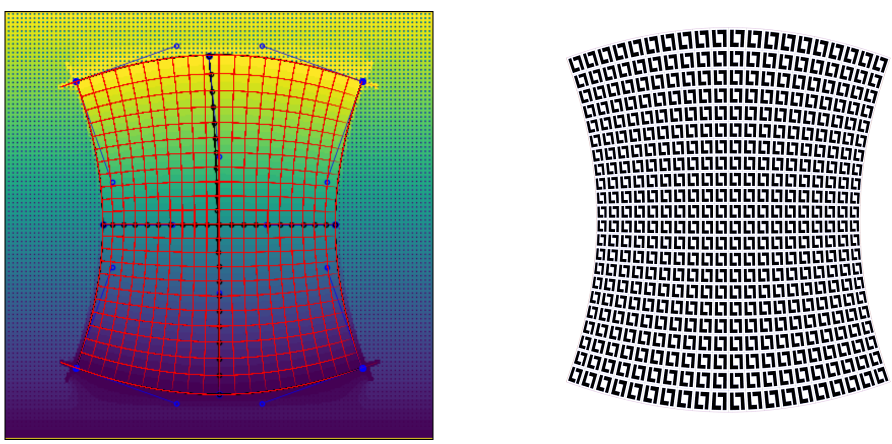
mathematical theorem which is so delightful that we shall leave the proof of it for one of your courses in mathematics. For any ‘ordinary function’ (..), the functions ψ and χ automatically satisfy the relations $\partial\psi/\partial x = \partial\chi/dy$, $\partial\chi/\partial x = -\partial\psi/\partial y$.” The χ is uniquely defined up to a constant if we know ψ. For computing equipotential lines and field lines, this constant will not be important in the end, so this does not change our results. In the same way in which ψ defines the distances between the equipotential lines, we should use the harmonic conjugate to find the distances between field lines.



(a) (b)

Figure 4: *Finding a grid inside a 2D capacitor. (a) color-coded potential ψ and field lines. (b) color-coded conjugate harmonic field χ with both field lines and equipotential lines.*

By way of illustration, we show the field lines inside a 2D capacitor in Figure 4. The upper plate is a conductor fixed at +1 volt whereas the lower plate is kept at 0 volt. The potential in the area between the plates is a real-valued function of x and y, denoted by $\psi(x+iy)$, which is color coded: yellow is the highest voltage (here 1) and dark purple is the lowest, green and blue being in-between values. The plate voltages act as boundary conditions and, in principle, we have enough information to solve Laplace’s equation. Once we have ψ, the field lines can be derived as we know the direction of electric field $E = -\text{grad}\,\psi$ where grad ψ has components $\partial\psi/\partial x$ and $\partial\psi/\partial y$. Thus we can find a field line of ψ from any desired starting point. Note the auxiliary line and the black dots in Figure 4, they form a “scaffold” that defines the points through which the field lines must pass.



(a) (b)

Figure 5: *From potentials to garments. (a) field lines and equipotential lines. (b) mapped motif.*

From this, we move to garment panels, Figure 5 shows the main idea. More implementation details can be found in the supplementary material. We just mention the ingredients by keywords: Bézier curves for user-defined contour lines, path integration for computing the conjugate harmonic, relaxation for solving the Laplace equation, Dirichlet and Neumann conditions for the horizontal and vertical contour lines, bilinear interpolation for mapping motifs in grid cells, and coding in Python (**`www.python.org`**). The code is available on **`github.com/LoeFeijs/HolomorphicMappings`**, but it is still very experimental.

Experimenting with the Holomorphic Mappings

When sewing panels together, conditions apply. First, the lengths of the sides should be equal. Next, the motif should match; for example, the upper and lower sides in Figure 5(b) match, as do the left and right sides. However, an upper line does not align with a left side (despite the fact that we use a motif whose symmetry is compatible with a 90° turn; the squeezing and stretching of tiles is different as can be seen in Figure 5b). One fashion idea was to create waves sewn along curved seams. To make the motif match, the scaffold line is placed near the edge[4] and its points positioned equidistantly, neglecting the conjugate potential (Figure 6). Although the left and right sides do not match well, we moved to garment construction (scale 1:4).

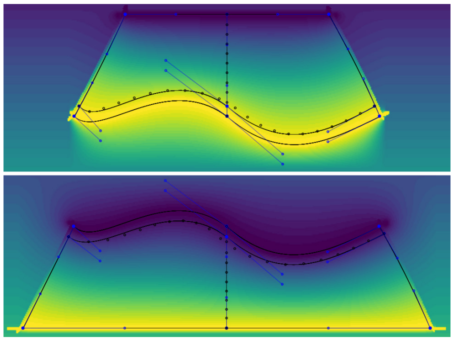


Figure 6: *Equidistant scaffolding.*

The overall design of the skirt is cone-like, yet with corners where the linear segments meet. The seam is highlighted by a strip of lace that stands out from the fabric (Figure 7).

Figure 7: *Wave skirt with morphed motif.*

In another experiment we created four-sided narrow panels, with different combinations of equidistant and non-equidistant scaffolding. Assembling five panels, we created a jacket at scale 1:4, see Figures 8 and 9.

[4] Ideally, the scaffold line should be on exactly on the edge. However, we found that the field line direction-finding is inaccurate near the edge, probably due to the pixelation effects of writing the boundary condition into an array. Our implementation constructs field lines by starting inside the boundary, making a final jump towards the edge by using the fact that the field line must be orthogonal to the edge (for which we have a polynomial formula).

Figure 8: *Narrow-panels based garment.*

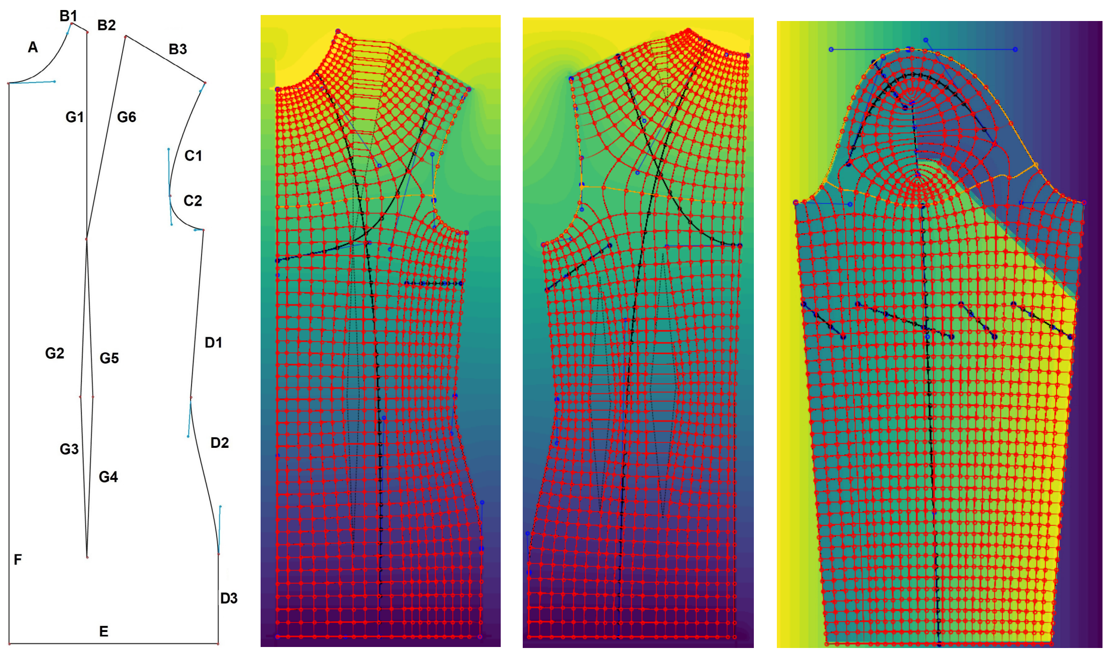
The design of the garments in Figures 7 and 8 was the most fun part of the project, because of the intense cooperation and interaction among the team members. While the tech-savvy Rong-Hao and Loe independently conceived the idea of using electric field lines and wrote hard-to-explain code, the fashion-savvy Holly and Marina envisioned new ways of creating garments, waves, mixed panel designs, 3D effects, and exciting sleeve-forms.

Figure 9: *Details of some (almost) matching seams.*

A Traditional Garment

We tested the basic pattern of Figure 10(a), used at Fashion Tech Farm (https://fashiontechfarm.com), encountering new challenges. First, the half-front panel does not have four sides but six for which we used three voltages (the other half-front is a copy). Next, there are darts, so we modified the relaxation algorithm to perform "worm holing" over the darts and the field line algorithm to jump across the darts. The sleeve is a four-sided panel, but its field lines do not match those of the body panels. Inspired by the notion of analytic continuation ([1], p.283), we added an extra conductor onto the sleeve (near the top, vertical). Computing the harmonic conjugate was confusing at first: our path integration yielded results that depended on the path. Then we remembered the residue theorem in complex analysis, which states that there is an increase of $2\pi i$ times a "residue" for each tour the path makes around a pole ([1], p.172). The sleeve of Figure 10(d) reveals the harmonic conjugate, and we observe the "date line" (diagonal, where the yellow and blue meet, rightmost on the sleeve). We had to stop the equipotential lines, otherwise they would run around the extra conductor forever. The scaffold lines are cut into more pieces than necessary, but this gave us some more fine-grained control over the field lines during experimentation.

This garment is not yet implemented; the design is stored away for future exploration. The open problems include the singularity on the sleeve's sewing line (where we encounter pentagonal grid cells). We may also require additional symmetries in the tessellation to get connecting motifs (like the double L motif of Figure 5(b) or the modified pied-de-poules with 180° rotational symmetry already used in Figure 8). For pied-de-poule, whose tiles have long tails, we must cut through the tiles anyhow — still we want the motifs to be whole again on the finished garment.



(a) (b) (c) (d)

Figure 10: *Computed grids for a full garment: (a) traditional garment panel. (b) front panel potential and grid, (c) back panel potential and grid, (d) sleeve with extra conductor, conjugate harmonic, grid.*

A Full-Size Garment

We used the cone-like mapping for the dress shown in Figure 11. The cone pattern is a 360° segment, i.e., a disk and we deploy the exponential function $w = e^{\omega z}$ as discussed earlier. The dress incorporates the red herring and Aruco code concepts already presented at Bridges 2024. The dress is a huge showpiece, showcased during Dutch Design Week and on the catwalk of the Next-Future Fashion show in Eindhoven.

Figure 11: *Red Herring dress with Aruco Codes and Pied-de-poule motif based on a seamless cone pattern (model Esmee Kobes, photographer Holly Krueger).*

Summary and Conclusions

The dress of Figure 11 is impressive by its size, and also interesting due to its potential role on the debate about ArUco codes and camera observations. To turn more general holomorphic mappings into fashion pieces is more adventurous — the adventure is still ongoing. The combined garment and motif geometries have an exciting appearance. At the same time, they illustrate the theory of analytic functions in an uncommon manner. *Acknowledgements:* We thank Troy Nachtigall and the Fashion Tech Farm and TU/e Wearable Senses communities for their support and the Bridges reviewers for their valuable feedback.

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