Unvexed Conformal Bodies: Musical Instruments, Ensembles, and Notations Derived from the Johnson Solids

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Abstract

This paper presents a new musical system derived from the geometry of the Johnson solids. Formal structure is used to design stringed and percussive instruments; ensembles are formed using a novel organization of the solids into "descendancies" based on the augmentation operation; and polyhedral nets are used as discrete-mathematical graphs through which compositions are notated as walks. The first performance using this system is also described.

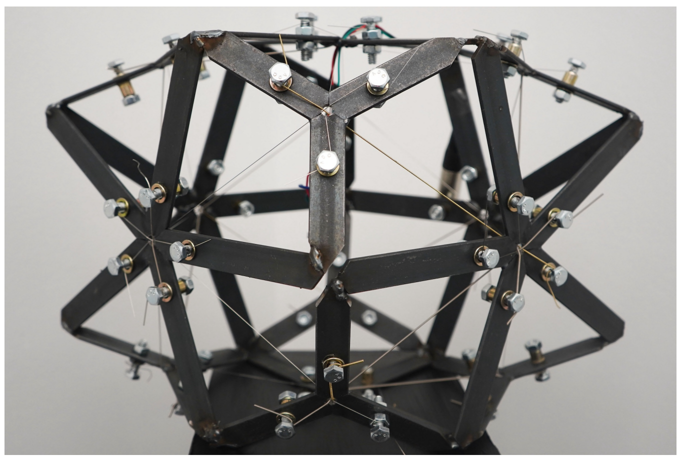


Figure 1: *The stringed instrument for the bilunabirotunda (J91).*

Introduction

In this work, I consider a *musical system* as a sonic framework with the following parameters: *instruments*, the sound-producing interfaces; *ensembles*, different ways to group the instruments; and *compositional notation*, the representation of temporal arrangements of instrumental sounds. For example, the musical system of Western European "classical" music has pianos, violins, clarinets, flutes, and so on; ensembles such as the string quartet, piano duet, and symphony orchestra; and the five-line staff system for notation.

Unvexed Conformal Bodies (UCB) is a new musical system derived from the Johnson solids, the set of all convex polyhedra with regular polygon faces. Here, I'll be using the generalized definition of "Johnson solid," which includes the 92 non-uniform polyhedra catalogued by Norman Johnson [2] as well as the 5 Platonic solids, the 13 Archimedean solids, and the infinitely many prisms and antiprisms.

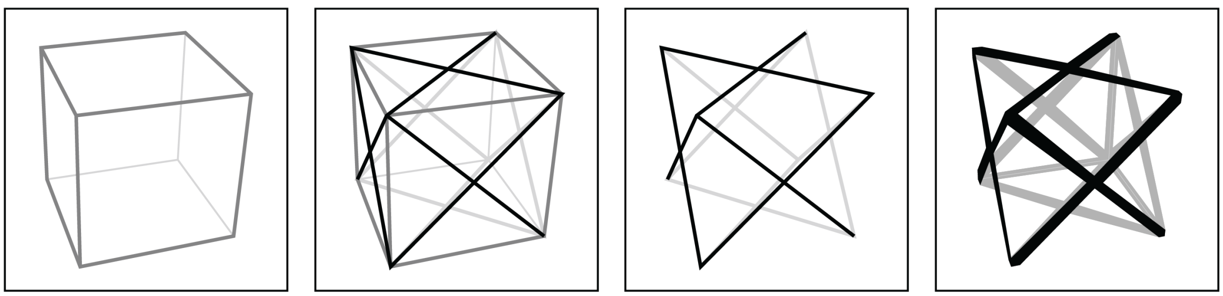
In the first section, I'll describe the instruments of *UCB*, which come in two types, *stringed* and *percussive*; each Johnson solid has a stringed instrument and a percussive instrument derived from its geometric structure. Then, I'll explain how ensembles are formed from a novel organization of the Johnson solids into "descendancies" based on the *augmentation* operation, the joining of one or more polyhedra to the face(s) of another. Last, I'll describe the compositional notation, which uses the polyhedral net of each instrument as a discrete-mathematical graph such that a composition is notated as a walk through that graph.

Finally, I'll describe the first iteration of *UCB*, which was held at Parsons School of Design in May 2023. I performed a composition titled *Body Feelings, Op. 1 No. 1*, a sonic walk through my memories of my first 1.5 years of gender transition. This first iteration also influenced the name of the work, *Unvexed Conformal Bodies*, as a nod towards unconventional embodiment through rearranging the descriptor of Johnson solids as "convex uniform bodies."

Instruments

Stringed Instruments

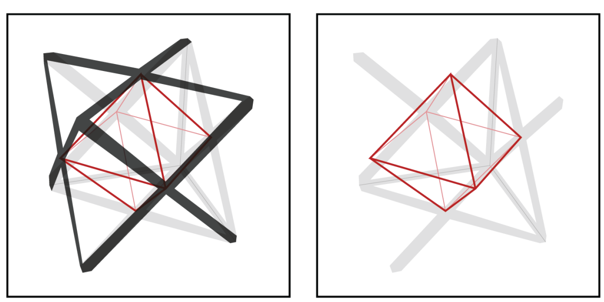
To find the structural form of each Johnson solid's stringed instrument, I first use Conway's *kis* operation [1], drawing new edges between the center of each face and each of the face's corners to create pinwheels. Then, the original edges of the polyhedron are erased. Finally, the pinwheel edges are thickened in the plane of the original face, creating a cage-like structure. Figure 2 shows this process on a cube.



(a) **(b)** **(c)** **(d)**

Figure 2: *Generating the stringed instrument structure for a cube: (a) original structure, (b) kis operation performed, (c) edges removed, (d) edges thickened.*

Strings are then strung between all adjacent pinwheel centers. The resulting polyhedron formed by these strings is the *dual* of the original solid, a geometric inversion of faces and vertices [4], as shown in Figure 3.



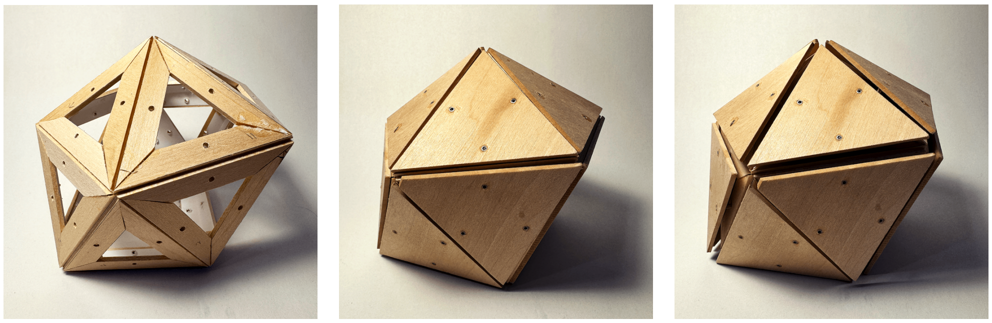
(a) **(b)**

Figure 3: *String placement for the cube: (a) strings inside the cage structure, (b) the dual revealed (the dual of a cube is an octahedron).*

The cage forms are built with MIG-welded, 3/4" wide 1/8" thick steel, providing structural integrity and resonance. The strings, a mix of electric guitar and electric bass strings, are attached with custom tuning pegs made of hex bolts, rivet nuts, and hex nuts. A completed stringed instrument for the *bilunabirotunda*, Johnson solid J91, is shown in Figure 1. This instrument is currently tuned by ear to non-specific frequencies in order to create a range of interesting dissonances.

Percussive Instruments

The percussive instruments are made of wood, with an underlying skeletal structure formed by thickening the edges of each face towards the center. These frames each have a corresponding lid which can be raised and lowered on pegs to decrease or increase resonance, thereby dampening or thickening the sound. The instrument is played by striking the center of each lid like a multi-headed drum, allowing the geometric structure of the Johnson solid to dictate their sounds, frequencies, and resonances. Figure 4 shows the percussive instrument of the *sphenocorona*, Johnson solid J86.



(a) **(b)** **(c)**

Figure 4: *The sphenocorona percussive instrument: (a) thickened skeletal structure, (b) lids attached and lowered, (c) lids partially raised.*

Ensemble Rules

Ensembles are made of one (1) stringed instrument supported by any number of percussive instruments. The stringed instrument may be picked from any of the Johnson solids, but the accompanying percussive instruments must be a *descendant* of the stringed instrument.

Descendancy is defined by the augmentation operation: the joining of one or more polyhedra to the face(s) of another. For example, Figure 5 shows that the *elongated square pyramid* (J8) is an augmentation of the *square pyramid* (J1) with the *cube* (Platonic solid). Thus, J8 is a descendant of both J1 and the cube.

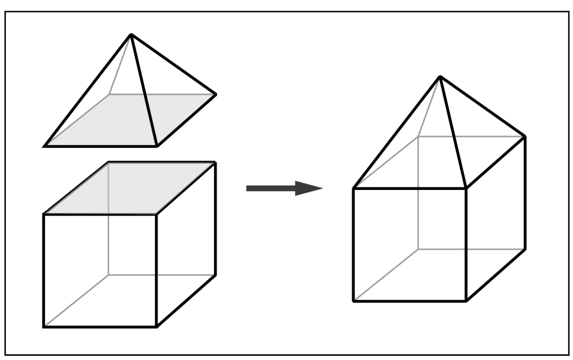


Figure 5: *The elongated square pyramid (J8) is an augmentation of the square pyramid (J1) with the cube.*

Johnson defines *elementary* solids as those which cannot be formed through augmentation; several Johnson solids are constructed as "cut-and-paste polyhedra" by putting elementary solids together [2]. In this novel organization, I categorize all existing augmentative relationships between Johnson solids, including those between non-elementary solids. For clarity, I've renamed elementary solids as *ancestors* and introduced terminology for *descendants*, which are solids formed through augmentation of ancestors. Ancestors are further categorized into 13 *pure* ancestors, which have no descendants, and 25 *non-pure* ancestors, which do have descendants. Descendancy directly from ancestors is called *primary lineage*. Some descendants also have their own descendants, which I'll call *extended descendants* within *secondary lineage*.

Table 1: *Primary Augmentative Lineages of Ancestors.*

Ancestor	Descendants
triangular prism	J7, J14, J26, J49-J51
square antiprism	J10, J17
pentagonal prism	J9, J16, J52, J53
pentagonal antiprism	*icosahedron* (P), J11
hexagonal prism	J18, J35, J36, J54-J57
hexagonal antiprism	J22, J44
octagonal prism	*rhombicuboctahedron* (A), J19, J37
octagonal antiprism	J23, J45
decagonal prism	J20, J21, J38-J43
decagonal antiprism	J24, J25, J46-J48
tetrahedron (P)	J7, J12, J14
cube (P)	J8, J15
dodecahedron (P)	J58-J61
truncated tetrahedron (A)	J65
truncated cube (A)	J66, J67
truncated dodecahedron (A)	J68-J71
square pyramid (J1)	*octahedron* (P), J8, J10, J15, J17, J49-J57, J87
pentagonal pyramid (J2)	*icosahedron* (P), J9, J11, J13, J16, J58-J61
triangular cupola (J3)	*cuboctahedron* (A), J18, J22, J35, J36, J44
square cupola (J4)	*rhombicuboctahedron* (A), J19, J23, J29, J37, J45
pentagonal cupola (J5)	*rhombicosidodecahedron* (A), J20, J24, J31-J33, J38-J41, J46, J47, J72-J82
pentagonal rotunda (J6)	*icosidodecahedron* (A), J21, J25, J32-J34, J40-J43, J47, J48
tridiminished icosahedron (J63)	*icosahedron* (P), J64
tridiminished rhombicosidodecahedron (J83)	*rhombicosidodecahedron* (A), J72-J82
sphenocorona (J86)	J87

Table 2: *Secondary Augmentative Lineages of Descendants.*

Descendant	Extended Descendants	Descendant	Extended Descendants
J7	J14	J55	J57
J8	J8	J56	J57
J9	J16	J58	J59-J61
J10	J17	J59	J61
J11	*icosahedron* (P)	J60	J61
J18	J35, J36	J62	*icosahedron* (P)
J19	*rhombicuboctahedron* (A), J37	J66	J67
J20	J38-J41	J68	J69-J71
J21	J40-J43	J69	J71
J22	J44	J70	J71
J23	J45	J76	*rhombicosidodecahedron* (A), J72
J24	J46, J47	J77	J72, J73
J25	J47, J48	J78	J72, J74
J49	J50, J51	J79	J74, J75
J50	J51	J80	*rhombicosidodecahedron* (A), J72, J73, J76, J77
J52	J53	J81	*rhombicosidodecahedron* (A), J72, J74, J76, J78
J54	J55-J57	J82	J72, J74, J75, J78, J79

These descendancy lineages are laid out in Tables 1 and 2, where I've listed polyhedra by prisms and antiprisms first, then Platonic solids (P), then Archimedean solids (A), and lastly Norman Johnson's 92 solids (J), using his numbering system with names omitted for brevity.

The remaining 13 pure ancestors are grouped into three special families: the Snub Family, the Pure-Archimedean Family, and the Lunar Family as shown in Table 3. (I've chosen not to count the infinite remaining prisms and antiprisms, even though they fit the definition of pure ancestor.) For polyhedra in these pure ancestor families, any polyhedron in their family can be used as an accompanying percussive instrument.

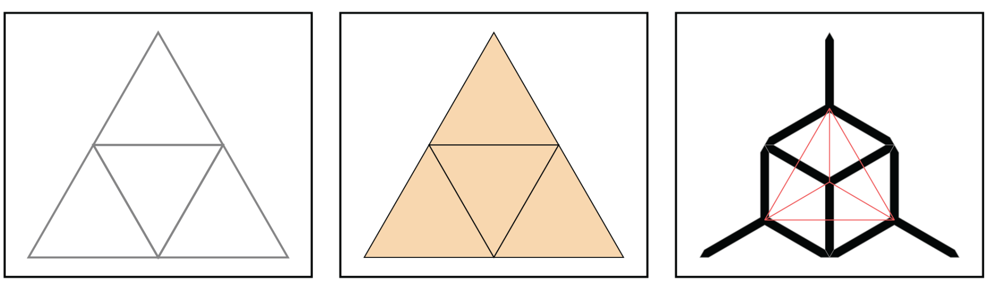
Table 3: *Pure Ancestor Families.*

Snub Family (4)	Pure-Archimedean Family (6)	Lunar Family (4+2)
snub cube	*snub cube*	*sphenocorona* (J86)
snub dodecahedron	*snub dodecahedron*	*augmented sphenocorona* (J87)
snub disphenoid (J84)	*truncated octahedron*	*hebesphenomegacorona* (J89)
snub square antiprism (J85)	*truncated cuboctahedron*	*disphenocingulum* (J90)
	truncated icosahedron	*bilunabirotunda* (J91)
	truncated icosidodecahedron	*triangular hebesphenorotunda* (J92)

The Snub Family includes the four polyhedra defined as snub in conventional namings [2]. Note that the Snub and Pure-Archimedean families share the *snub cube* and *snub dodecahedron*, highlighted in gray in Table 3. Note also that the Lunar Family includes the *sphenocorona* (J86), a non-pure ancestor, and the *augmented sphenocorona* (J87), the sole descendant of J86 when augmented by a *square pyramid* (J1), as highlighted in red in Table 3. J86 and J87 are included in the Lunar Family as they are also built from *lunes*, a square with two triangles on either end.

Notation and Composition

Every polyhedron has an associated *net*, an unfolded version of its faces. For example, Figure 6a shows the net of a *tetrahedron* (Platonic solid). Each instrument in any given ensemble can also be unfolded into a net. A percussive instrument's net looks exactly the same as their solid's, and a stringed instrument's net is simply a pinwheeled version of their solid's, as shown in Figures 6b and 6c.



(a) **(b)** **(c)**

Figure 6: *Examples of nets: (a) the net of a tetrahedron, (b) the net of the tetrahedron's percussive instrument, (c) the net of the tetrahedron's stringed instrument.*

The purpose of *notation* is to record the temporal arrangement of instrumental sounds. Percussive instruments can be played by striking any of their faces, and stringed instruments can be played by plucking any of the strings running between faces. The nets of these instruments can therefore be used to as a tool record the composer's sequential choices by treating them as discrete-mathematical graphs.

For percussive instruments, consider the center of each face as a vertex of a graph, and make it complete by connecting every vertex to every other vertex. Then, the set of all possible compositions for this instrument–all possible sequences in which to strike its faces–is represented by the set of all possible walks through this graph, with the vertices on each walk corresponding to face-strikes. (We can account for repeated strikes by allowing self-loops in the graph). For stringed instruments, this relationship between compositions and walks is equally true; the only difference is that *edges* on each walk correspond to string-plucks.

The *net-graphs* of each instrument thus provide a kind of blank "staff" upon which compositions may be visually notated by drawing walks. Within an ensemble, these walks may happen *concurrently*, with instruments being played at the same time; or they may happen *sequentially*, with a walk starting in one instrument's net-graph before jumping to the net-graph of another; and of course, concurrent and sequential walks may be used in various combinations to create a multilayered composition.

Figure 7 shows notation of a simple composition, sequenced arbitrarily, for a *tetrahedron* stringed instrument and *triangular bipyramid* percussive instrument (J12), with the strings numbered and the faces labelled alphabetically for clarity. The notated walk through these net-graphs represents the following sequence of string-plucks and face-strikes: [G–2–6–I–K–K–L].

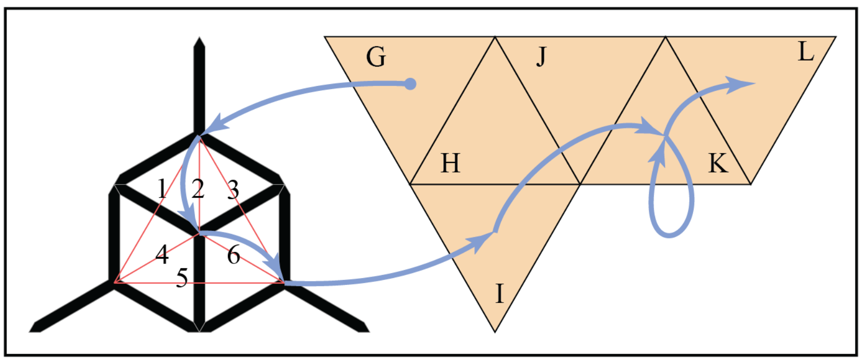


Figure 7: *An example of a notated composition for a tetrahedron stringed instrument and triangular bipyramid percussive instrument.*

By associating vertices of the graphs with external material, the composer can also create a narrative sequence alongside the sonic sequence; for instance, if Figure 7 had words assigned instead of letters, a composition would also create a poem. This technique can be used with sentences, photographs, objects, or anything else to create a narrative alongside the sound, as I demonstrate in the next section.

Implementation and Performance

For the first iteration of *UCB*, I chose a Lunar family ensemble: *bilunabirotunda* stringed instrument (J91), *sphenocorona* percussive instrument (J86), and *augmented sphenocorona* percussive instrument (J87). I created a notation-quilt with the instruments' net-graphs appliqued on top, as shown in Figure 8. For my compositional material, I chose diary entries and bodily photos representing my first 1.5 years of gender transition. Diary entries were written onto faces of the percussive instruments and photos were attached to vertices on the stringed instrument, with thematically related materials placed adjacently in the net-graph.

Instead of representing the composition directly on the notation-quilt, I wrote out its sequence in words. Each percussive face's writing was distilled to a representative phrase and each pinwheel's picture was described with a phrase. I used these shortened phrases to create a verbal representation of the composition, shown in Figure 9, such that each chunk of text represents a face-strike or string-pluck. This text was distributed as a "program" representing the score of the piece.

Figure 8: *The notation-quilt for Body Feelings, Op. 1 No. 1, with written diary entries on percussive faces and bodily photos at pinwheel centers.*

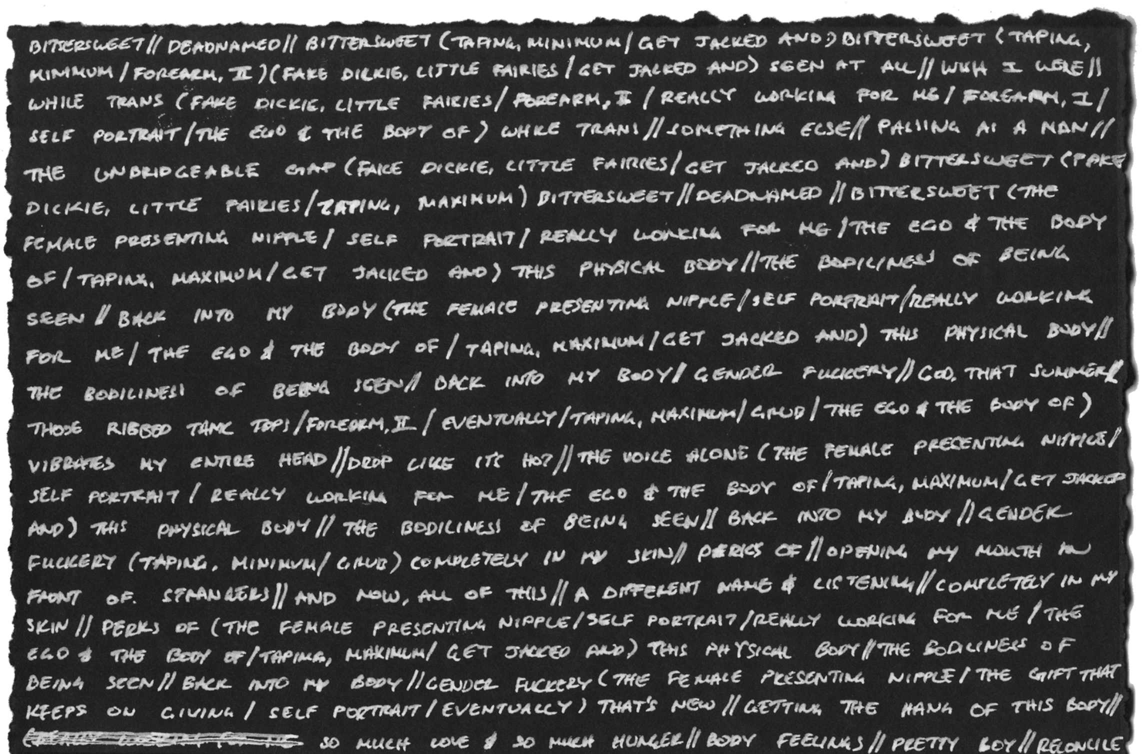


Figure 9: *The verbal representation of the compositional sequence in Body Feelings, Op. 1 No. 1.*

This composition, titled *Body Feelings, Op. 1 No. 1*, therefore represented a new understanding of my lived experience of transition, revisiting and reordering specific memories through performative sonic sequencing. The piece was performed four times in May 2023 at Parsons School of Design, as shown in Figure 10. A performance video and additional images are available online [3].

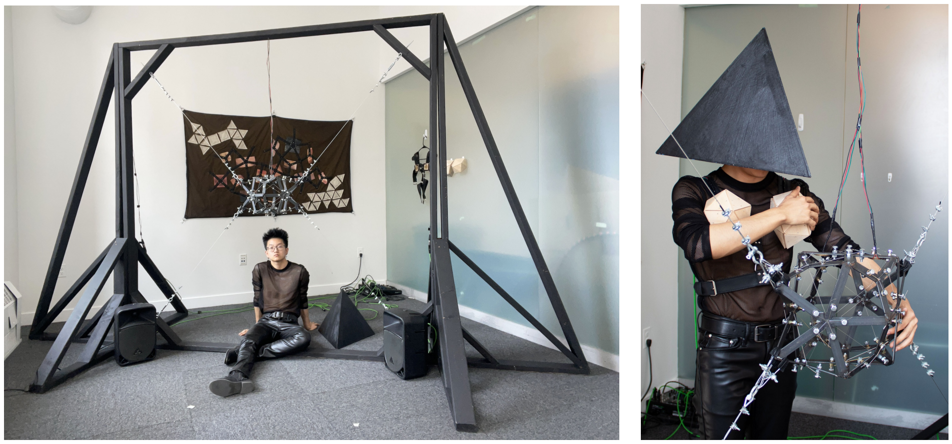


Figure 10: *The performance of Body Feelings, Op. 1 No. 1 at Parsons School of Design.*

Conclusions and Future Work

This new system of instruments, ensemble rules, and compositional notations derived from the Johnson solids provides a robust yet flexible environment for performative musical exploration. In *Body Feelings, Op. 1 No. 1* described above, the compositional material revolved around extremely personal reflections on embodiment, which influenced my aesthetic choices of instrument placement, costuming, and choreography. Future iterations may take a variety of aesthetic presentations depending on the compositional subject.

Additionally, there is room for development of the system itself. One area that deserves more thought is the tuning of the stringed instruments, as there are many geometric ratios within the Johnson solids that may be mapped into tuning rules. These new tunings could then be used to create a new music theory of polyhedral harmony and melody. This theory could also define adjacency-based rules for composition, derived from the net-graphs. Finally, the net-graphs could be reframed as visual representations of a composition rather than as functional notation, as longer compositions quickly become unreadable when notated on a net-graph.

Acknowledgements

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