Artfully Modeling Hyperbolic Planes through Tessellations

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Abstract

In this workshop, we will take three different examples of hyperbolic artworks, which are modeled using the two-dimensional Poincaré disk, and lift them off the page into our three-dimensional world. The tessellated artwork will be deconstructed, slightly modified, and then reassembled. With the addition of each new piece, the three-dimensional art will slowly obtain the negative curvature seen in hyperbolic geometry making it twist and curl. We will be using Dániel Erdély's Hyperbolic *Spidrons 55*, and Doug Dunham's *Fish (5,4)* and *Butterflies (8,3)* as the artwork throughout this workshop.

Introduction

A tessellation is a covering of a surface with shapes that do not overlap and contains no gaps. There are only three regular polygons that tessellate the Euclidean plane: triangles, squares, and hexagons. To create a tessellation with triangles in the Euclidean plane, we must join six triangles at each vertex. Similarly, we can tessellate using four squares or three hexagons at each vertex. The Euclidean plane has zero curvature, meaning the plane is flat and thus can be tessellated with regular polygons.

In contrast, hyperbolic planes have negative curvature, resulting in the plane "folding in" on itself—think kale leaves or coral. It seems that hyperbolic geometry is a never-ending source of inspiration when it comes to connecting mathematics and art. Many artistic renditions of the Poincaré disk model have been generated over the years, with some of the most recognizable pieces being done by M.C. Escher [3]. Even with just a quick look through the Bridges Archives, one finds three-dimensional models of the hyperbolic plane through crochet, which are inspired by D. Taimina [8][9], and circular looping [7] or a two-dimensional investigation through tessellations [1][5], gaming inspiration [6], or even embroidery [2].

This workshop will connect the two-dimensional and three-dimensional views of hyperbolic geometry. We will begin with hyperbolic tessellations modeled through the Poincaré disk, which is a two-dimensional representation of the hyperbolic plane (see Figure 1). There are distortions in this two-dimensional model since it must represent the negative curvature of the hyperbolic plane on the Euclidean plane. We will tape modified pieces of the Poincaré disk tessellations together to showcase the tessellation in three-dimensions. Since the hyperbolic plane has negative curvature, these three-dimensional models will as well—the larger the artwork becomes the more curvature will appear.

(a)   (b)
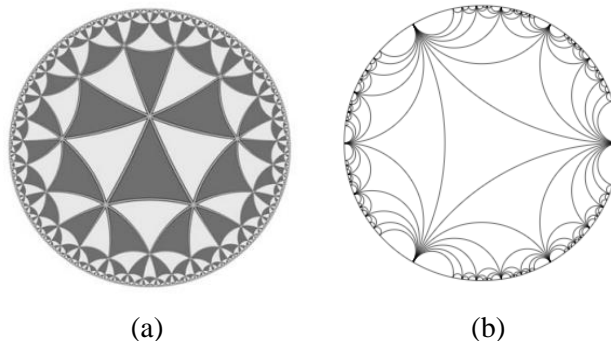
Figure 1: *Poincaré disk models with (a) triangle tessellations [4] and (b) hyperbolic lines [10].*

Recall, that when tessellating on the Euclidean plane, six triangles meet at each vertex. The hyperbolic plane is not limited to only six meeting at a point. Figure 1(a) has eight (there also can be seven) triangles meeting; however, the triangles must be distorted to represent the hyperbolic plane on the Euclidean plane.

The primary goal of this workshop is to bring artistic tessellations of the Poincaré disk into three-dimensions to illuminate the curvature of the hyperbolic plane. To accomplish this, the participants will be guided through the recreation of many of these artworks in three-dimensions. We will be utilizing the artwork of Dániel Erdély, *Spidron 55*, and Doug Dunham, *Fish (5,4)*, and *Butterflies (8,3)* throughout.

Workshop Outline

This 90-minute workshop will include three main parts:

Part 1. Prelude (20 minutes)

At the start of the workshop, we discuss background information as well as begin working with paper and tape to create a hyperbolic plane using equilateral triangles.

Part 2. Assembling Hyperbolic Art (60 minutes)

This is the main creative portion of the workshop. Prior to reconstructing the hyperbolic art, we discuss why the pieces from Erdély's and Dunham's original art need to be modified to create the new three-dimensional model (see Figure 2). Here, participants use the modified pieces from *Spidron 55, Fish (5,4)* and *Butterflies (8,3)* to create a three-dimensional model of hyperbolic art.

Part 3. Discussion (10 minutes)

To conclude, participants share their completed art products and discuss the differences between various models. In addition, we share ideas to transform this workshop for use in the classroom.

Materials and Instructions

Materials

Each participant is provided with images of *Spidron 55, Fish (5,4),* and *Butterflies (8,3)* (Figure 2), scissors, tape, equilateral triangles, and all pieces needed for each artwork (see Figure 6 for the individual pieces used).

(a) (b) (c)
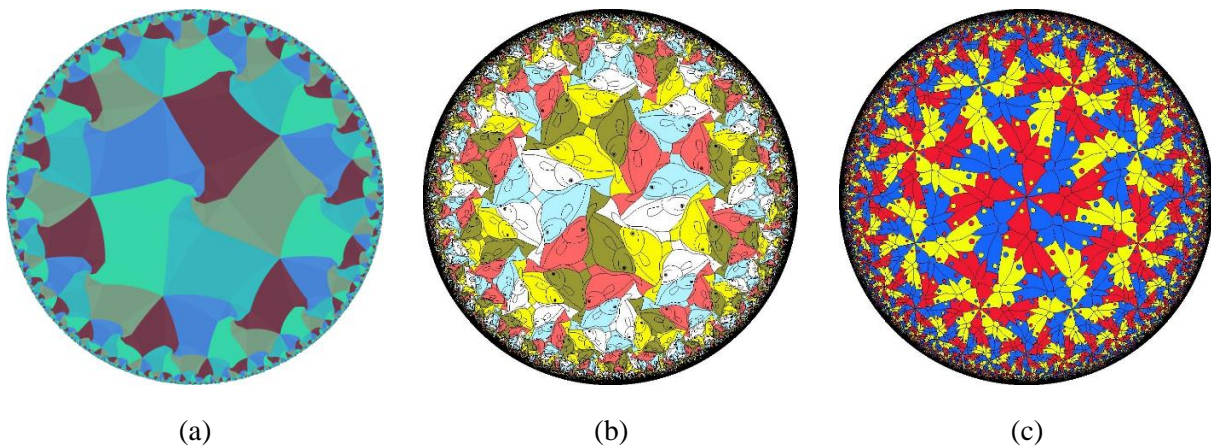
Figure 2: *Original artwork: (a) Spidron 55, (b) Fish (5,4), (c) Butterflies (8,3).*

Instructions

Workshop Part 1. Prelude: When tessellating the Euclidean plane with a regular polygon, there are only three possibilities: triangles, squares, and hexagons—all polygons will be considered regular unless otherwise stated. Tessellating with polygons on the Euclidean plane takes six triangles, four squares, or three hexagons meeting at each vertex (see Figure 3). Hyperbolic geometry uses more polygons; in Figure 4 we use seven triangles, five squares, or four hexagons; however, more may be used if desired, e.g. Figure 1(a) uses eight triangles.
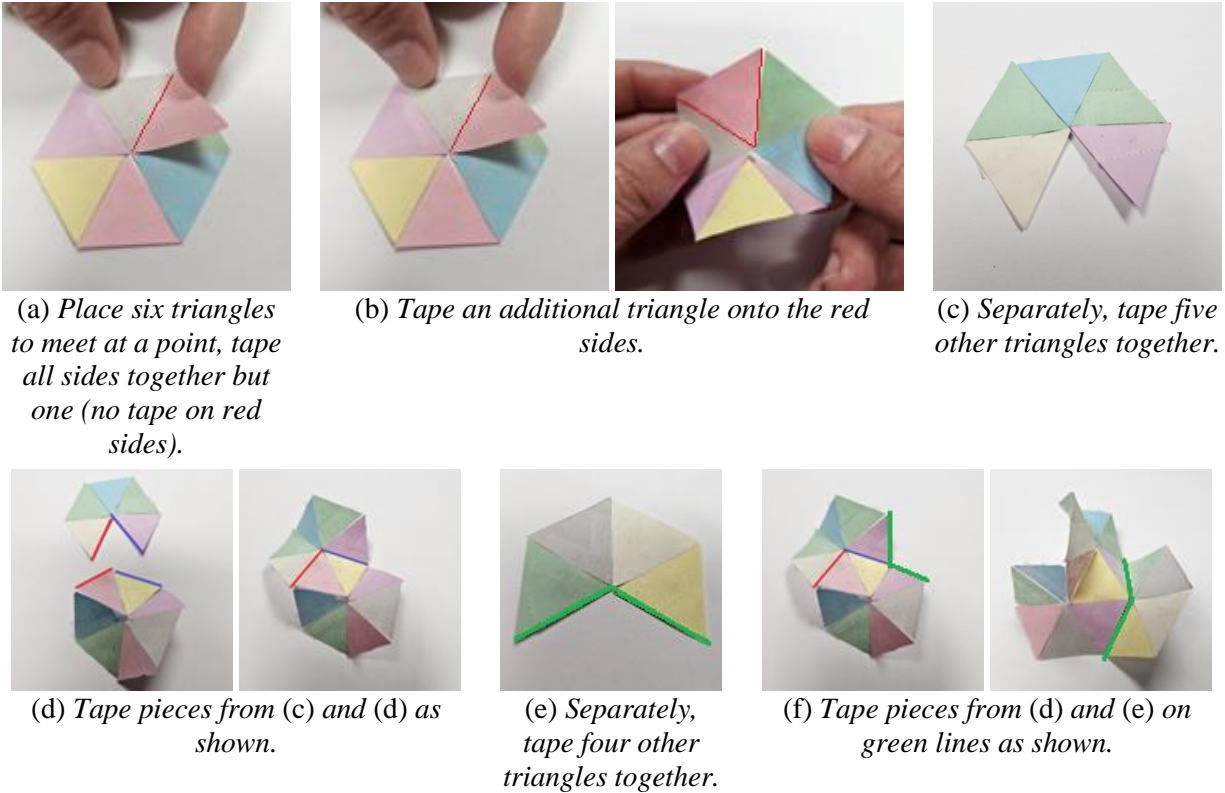

(a)   (b)   (c)
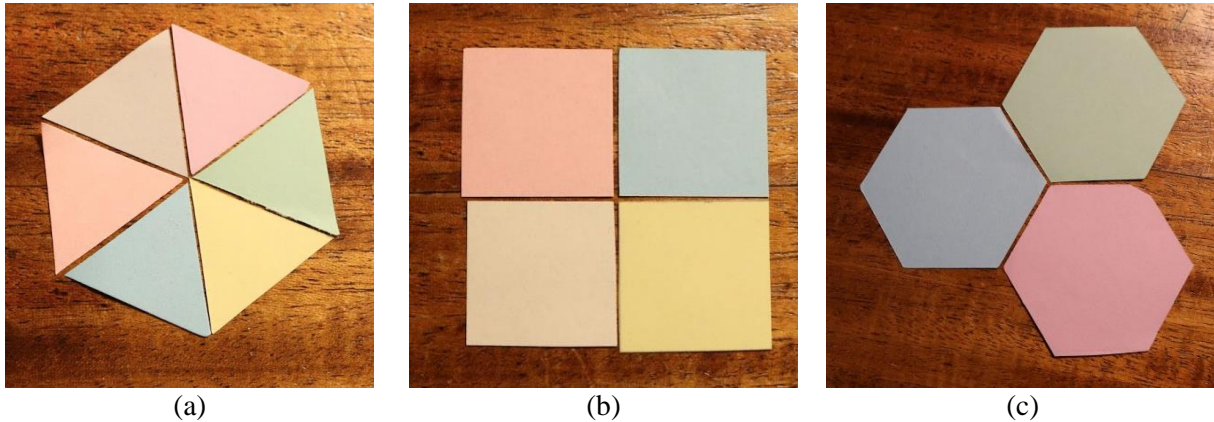
Figure 3: *Tessellations in the Euclidean plane: (a) six triangles, (b) four squares, (c) three hexagons.*

When an additional triangle, square, or hexagon is added at the vertex; the flat tilings, seen in Figure 3, become curved. Notice in Figures 4(a)-(c) the tiling rises differently for each polygon; this is due to the additional angle that is added to the point where all vertices meet. All of Figure 3's polygons meet at a point, and the sum of their angles is 360°. Figure 4(a)-(c) includes an additional polygon, so now the sum of the angles is 420°, 450°, and 480°, respectively. The larger the sum of the angles at a point, the more drastic the rise becomes.

(a)   (b)   (c)   (d)
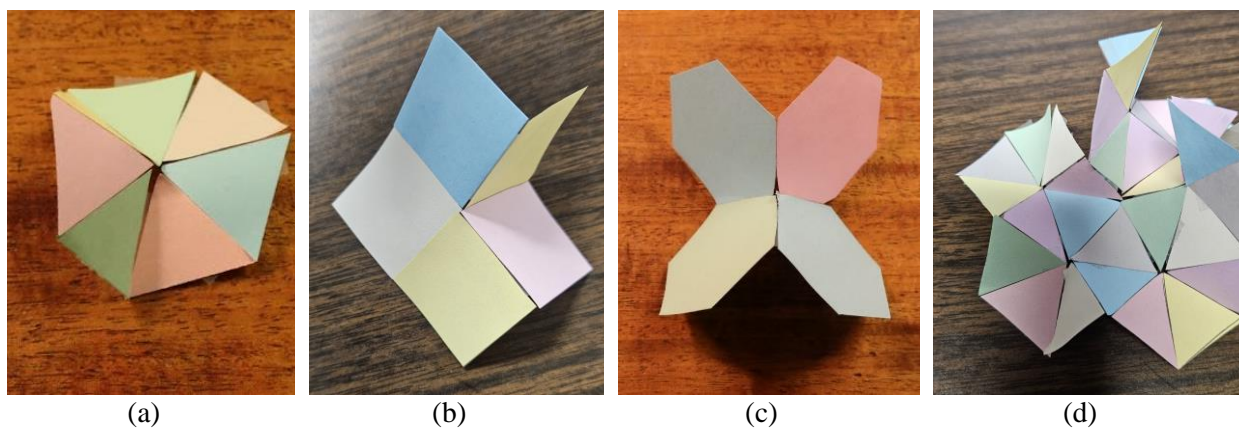
Figure 4: *Modification of Euclidean tessellations using: (a) seven triangles, (b) five squares, (c) four hexagons, and (d) expanded seven-triangle tessellation.*

The participants will use equilateral triangles to create Figure 4(d) by following the instructions in Figure 5. Once steps in Figure 5(a)-(f) are completed, participants will repeat the step in Figure 5(f), surrounding the object made in Figure 5(b). By repeating this process, the participants will create a three-dimensional model of the hyperbolic plane, as in Figure 4(d).

(a) *Place six triangles to meet at a point, tape all sides together but one (no tape on red sides).*

(b) *Tape an additional triangle onto the red sides.*

(c) *Separately, tape five other triangles together.*

(d) *Tape pieces from* (c) *and* (d) *as shown.*

(e) *Separately, tape four other triangles together.*

(f) *Tape pieces from* (d) *and* (e) *on green lines as shown.*

**Figure 5:** *Creating a seven-triangle tessellation.*

Creating the seven-triangle tessellation gives a basic understanding of the process that is used when constructing more elaborate examples, along with practice working with the negative curvature of the hyperbolic plane.

Workshop Part 2. Assembling Hyperbolic Art: Before assembling the hyperbolic models, we first showcase the differences between the elements of the original artworks and the modified pieces, as depicted in Figure 6. These modifications are necessary because when we represent the hyperbolic plane on the Euclidean plane, as exemplified by the Poincaré disk model, a certain level of distortion is inevitable. Specifically, distortions of size and "straightness" are observed.

The representation of the Poincaré disk in Figure 1(a) shows distortions of size—the triangles appear to be different sizes. It is important to recognize that on a true hyperbolic plane, the actual size of these pieces does not change. The pieces only appear to get smaller in the Poincaré disk so that the hyperbolic image can lay flat and fit within a circle on the Euclidean plane.

The concept of "straightness" in a hyperbolic plane is also worth taking into consideration. For example, in Figure 1(b), the "curved" lines spanning across the circle are not curved, these are "straight" lines on the hyperbolic plane. This distortion of lines is a consequence of how the negative curvature of the hyperbolic plane is being modeled on a two-dimensional Euclidean plane. On a hyperbolic plane, these lines would be indeed straight.

Combining these two ideas, we are able to modify pieces within Poincaré models to more closely resemble what the elements would look like on the hyperbolic plane. Figure 6 shows the original pieces, along with the modified pieces, respectively. To modify the pieces to be efficiently used in the three-dimensional art, all pieces need to be the same size, as well as having some of the "curved" sides "straightened."
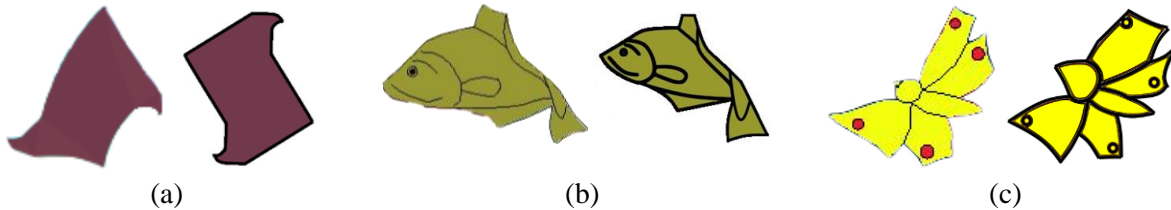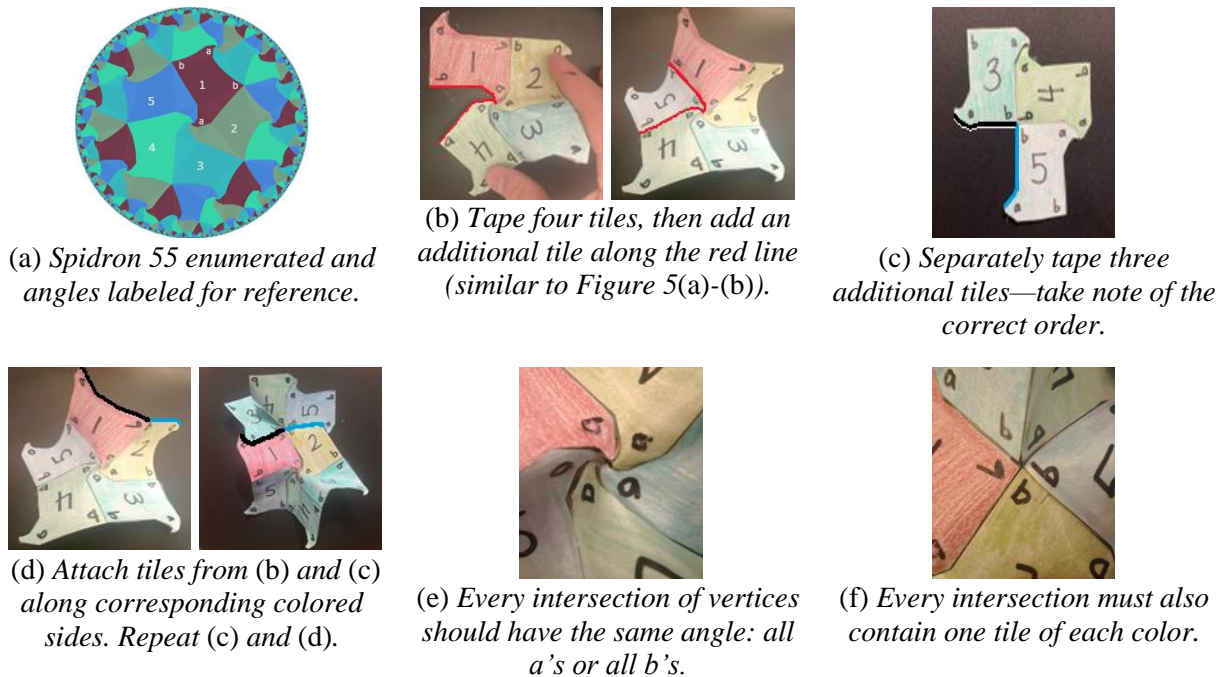
Figure 6: *Unmodified/Modified pieces from (a) Spidron 55, (b) Fish (5,4), (c) Butterflies (8,3).*

The participants have multiple copies and colors of the modified pieces of Erdély's and Dunham's works to tape together. From here on we will refer to the individual pieces as tiles.

The rest of this subsection will include directions for creating the three-dimensional artwork for *Spidron 55*, *Fish (5,4)*, and *Butterflies (8,3)*. However, a few comments to ease the construction:
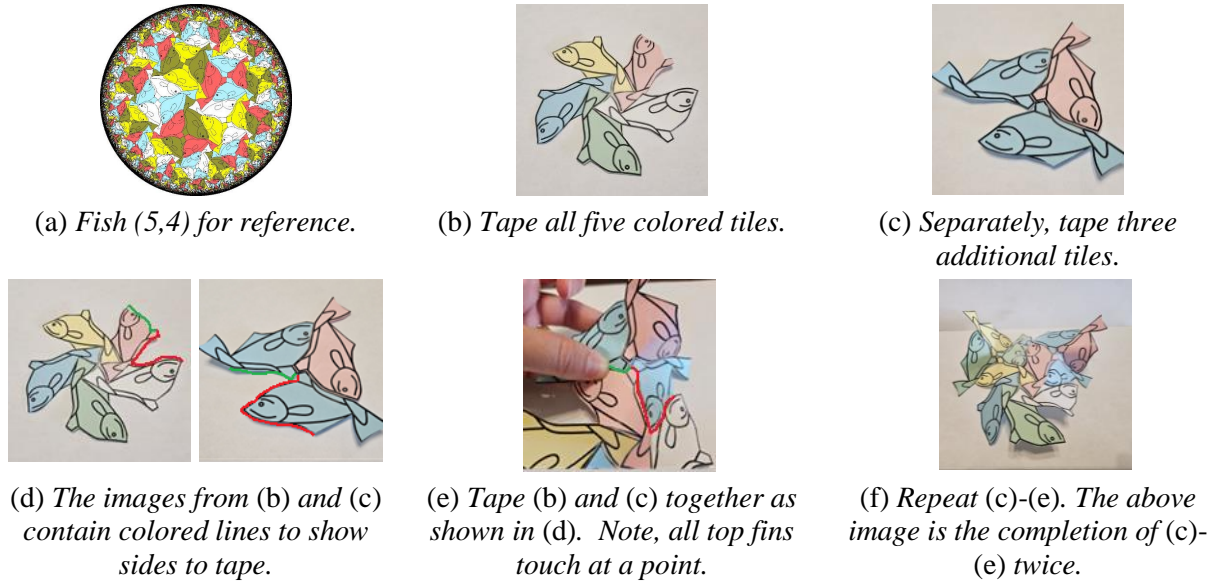
- Using smaller pieces of tape to attach tiles helps with bending and moving the tiles to fit.

- With each attachment of a new set of tiles, the artwork has more curvature, thus having an extra pair of hands (or set of fingers) can be helpful to bend and fold the artwork as needed.

We now state the directions for the three-dimensional artwork inspired by *Spidron 55* in Figure 7. The provided modified tile from Figure 6(a) is not enumerated, nor contains labeled corners; however, for ease of direction, we have included these additions for Figure 7. Note, when proceeding, each tile has two different angles: the right angles labeled 'b,' and the spiral angles labeled 'a.' Additionally, it is recommended to position the tiles in the same order as *Spidron 55*. After the steps in Figure 7(b)-(d) are completed, repeat the steps in Figure 7(c) and (d) to create the full artwork shown in Figure 10(a).
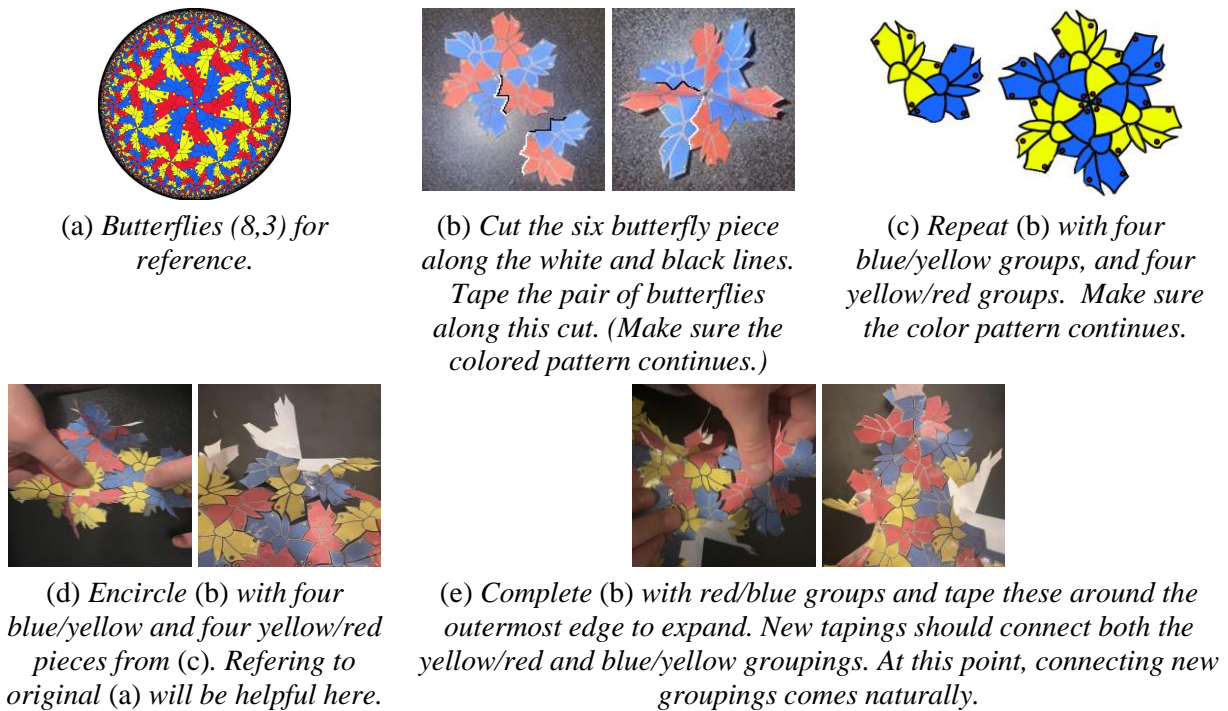

(a) *Spidron 55 enumerated and angles labeled for reference.*

(b) *Tape four tiles, then add an additional tile along the red line (similar to Figure 5(a)-(b)).*

(c) *Separately tape three additional tiles—take note of the correct order.*

(d) *Attach tiles from (b) and (c) along corresponding colored sides. Repeat (c) and (d).*

(e) *Every intersection of vertices should have the same angle: all a's or all b's.*

(f) *Every intersection must also contain one tile of each color.*

Figure 7: *Creating the three-dimensional Spidron 55 artwork.*

Creating the three-dimensional *Fish (5,4)* is more difficult as the fish tiles have more curvature than the *Spidron 55* tiles, so it is suggested to be the second artwork created. See Figure 8 for directions, and Figure 10(b) for the final product. Figure 8(f) shows how quickly the curvature develops when creating this three-dimensional art.

(a) *Fish (5,4) for reference.*

(b) *Tape all five colored tiles.*

(c) *Separately, tape three additional tiles.*

(d) *The images from* (b) *and* (c) *contain colored lines to show sides to tape.*

(e) *Tape* (b) *and* (c) *together as shown in* (d)*. Note, all top fins touch at a point.*

(f) *Repeat* (c)-(e)*. The above image is the completion of* (c)-(e) *twice.*

Figure 8: *Creating the three-dimensional Fish (5,4) artwork.*

When creating the three-dimensional artwork of *Spidron 55* and *Fish (5,4)*, each tile was added individually. However, as another approach, we can initially group tiles together and strategically cut and tape additions, making the artwork grow more quickly. While this method does speed construction, some of the complexity of the hyperbolic plane can be lost in this approach. For this reason, we are introducing this method last. What follows in Figure 9 are the directions for constructing *Butterflies (8,3)* using this alternative approach. One can just as easily construct the artwork by attaching each butterfly tile individually by following similar methods as *Spidron 55* and *Fish (5,4)*.

(a) *Butterflies (8,3) for reference.*

(b) *Cut the six butterfly piece along the white and black lines. Tape the pair of butterflies along this cut. (Make sure the colored pattern continues.)*

(c) *Repeat* (b) *with four blue/yellow groups, and four yellow/red groups. Make sure the color pattern continues.*

(d) *Encircle* (b) *with four blue/yellow and four yellow/red pieces from* (c)*. Refering to original* (a) *will be helpful here.*

(e) *Complete* (b) *with red/blue groups and tape these around the outermost edge to expand. New tapings should connect both the yellow/red and blue/yellow groupings. At this point, connecting new groupings comes naturally.*

Figure 9: *Creating the three-dimensional Butterflies (8,3) artwork.*

When more tiles are used in each art, the more ruffled the three-dimensional artwork becomes. Figure 10 shows the artwork. One may expand as far as desired, increasing the curvature with each new addition.
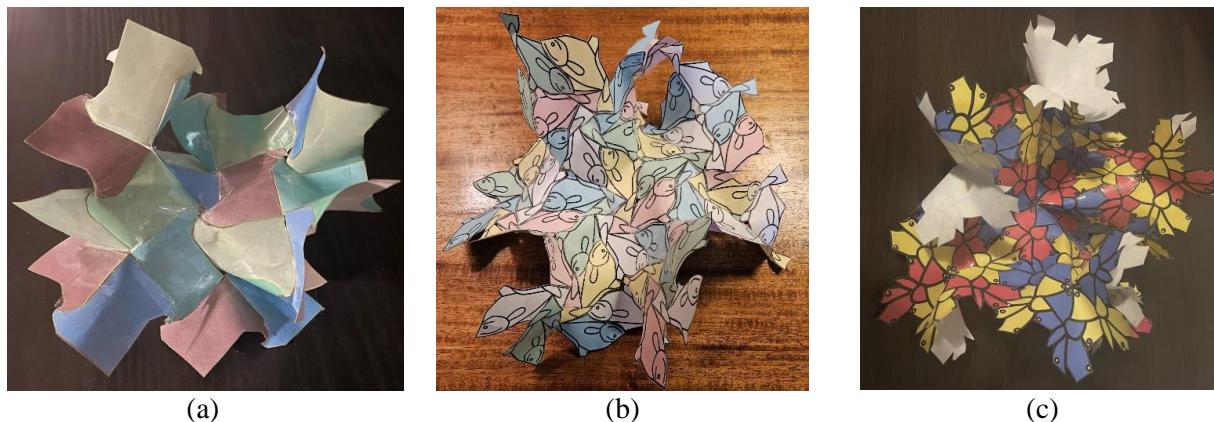
Figure 10: *New three-dimensional art: (a) Spidron 55, (b) Fish(5,4), (c) Butterflies(8,3).*

Workshop Part 3. Discussion: To conclude the workshop we rejoin as a group to discuss the activities. We focus on two main concepts: the artwork and three-dimensional creations, as well as adapting this workshop into a wider classroom setting.

After working through the four tessellations it is clear there are differences among them:

- Triangles: There is no need to focus on different colors or patterns. Yet, it can be easy to extend in one direction, and not outward from the first grouping of seven-triangles.

- *Spidron 55*: Tiles can be viewed as having four "mostly" straight sides, so attaching each tile can be completed with one larger piece of tape. However, the spiral corner can take time to attach.

- *Fish (5,4)*: Tiles have more curvature, therefore, when taping together one must curve and bend the tiles more than the previous two tessellations. In addition, for the first step, shown in Figure 8(b), all the fish tails meet and the center stays flat on the table, no other tessellations start off with zero curvature.

- *Butterflies (8,3)*: Directions are presented as taping groups of tiles, not individual tiles—an advantage of this is that there are eight butterflies attached at the center vertex, and having a group of six shortens the amount of time taping.

To adapt the material from a 90-minute session to multiple days in a classroom one can expand and investigate individual parts of the workshop:

- The workshop's main focus is converting two-dimensional artwork into three-dimensional artwork; however, a discussion on the summation of angles as well as tessellating and exploring the difference between triangles and other regular polygons can be a great study.

- A nice relationship between the two-dimensional Poincaré disk models and the three-dimensional artwork is seen when creating a larger artwork. By creating larger artwork more tiles are added and thus will produce more curvature. To represent this three-dimensional artwork in a two-dimensional way, the tiles closer to the boundary must be distorted and shrunk to fit flat in two-dimensions.

- The creation process can be quickened by using the *Butterflies (8,3)* method and grouping tiles together, so students insert tiles into groupings, not tape individual tiles together.

- Tailoring the constructions to the maturity of the audience: the seven-triangle tessellation and *Spidron 55* artwork are recommended for younger audiences. For an older audience work on the *Fish (5,4)* or *Butterflies (8,3)* artworks.

We encourage others to come up with their own observations of differences between the artwork and modifications to the creations within a classroom setting.

Conclusion

This workshop begins by leading participants through the construction of a three-dimensional representation of a rather simple tessellation—in fact, a tessellation many elementary school students have probably previously encountered. As we progressed, we introduced progressively more intricate tessellations allowing the participants to develop an understanding of the complexities of the hyperbolic plane in a unique and interesting way. This novel concept regarding the curvature presented in the hyperbolic plane is one that not even many university mathematics majors encounter in their studies.

Art is a fantastic tool that can be used to educate about the vast and puzzling world of hyperbolic geometry. By recreating two-dimensional representations of hyperbolic geometry artwork, and showcasing it in the third dimension, one can discover the properties and intricacies of hyperbolic planes.

Acknowledgements

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