

Creating Square Koch Surfaces Using Origami

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Abstract

In this workshop, we will learn how to construct Square Koch Surfaces. Using the five modules I designed, we will build and assemble iterations of Square Koch Surfaces. We will also talk about extending these surfaces into a Square Koch Cube, gaining a deeper understanding of the intricacy and beauty of fractal geometry.

Introduction

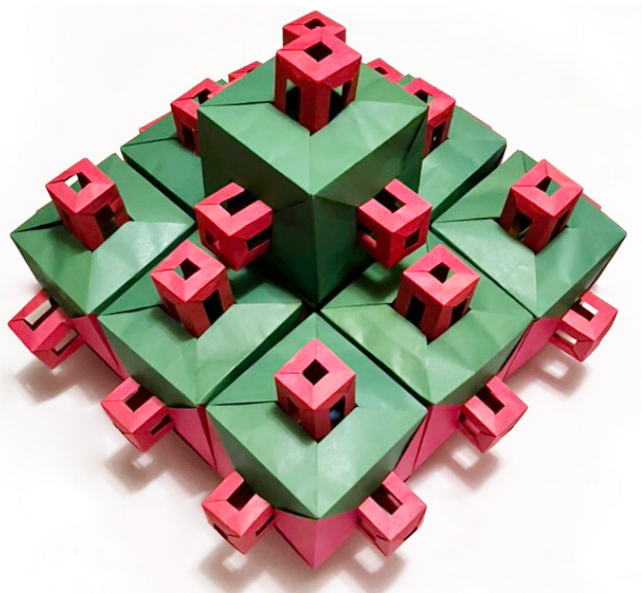


Figure 1: *Three-Dimensional Square Koch Surface, using origami.*

In 1904, Helge von Koch introduced the *Koch Curve*. To construct this curve, begin with a line segment and divide it into three equal sections. Then, replace the middle section with two sides of an equilateral triangle (Figure 2(a)). This procedure is repeated on each of the four equal line segments created in the previous step. This process is repeated an infinite number of times to create the Koch Curve. A *Square Koch Curve* replaces the middle segment of each line with three sides of a square instead of two sides of a triangle (Figure 2(b) and 2(c)).

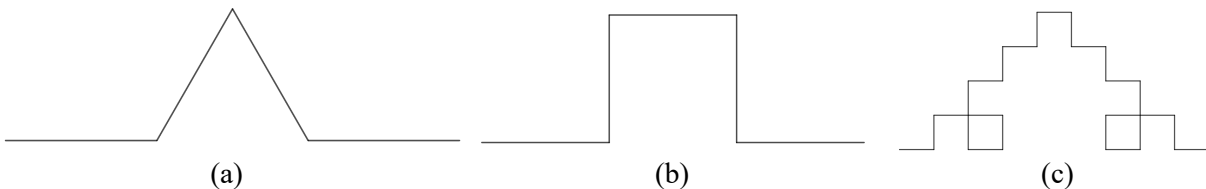


Figure 2: *a) Koch Curve iteration 1, b) Square Koch Curve iteration 1, c) Square Koch Curve iteration 2.*

To make a *Koch Surface*, begin with a triangle. Connect the midpoints of the three sides of the triangle, creating an outline of a triangle with a quarter of the area of the starting triangle. Replace the center triangle with three sides of a tetrahedron (Figure 3(a)). This process is then repeated on each of the six triangles just formed. This process is iterated to form the Koch Surface. A *Square Koch Surface* is formed by a similar operation beginning with a square: the square is divided into nine squares, each with one-ninth the area of the starting square. The center square is removed and replaced with five sides of a cube (Figure 3(b)). Then, the construction is repeated on each of the thirteen new squares. Figure 3(c) shows a second iteration Square Koch Surface. Repeating this process infinitely forms the Square Koch Surface [1]. Figure 1 represents a second iteration of a Square Koch Surface. I'd like to note that the green portion of the model represents the Square Koch Surface, and the edges are a slim section of four more Square Koch Surfaces that would be visible if it were extended into a *Square Koch Cube*.

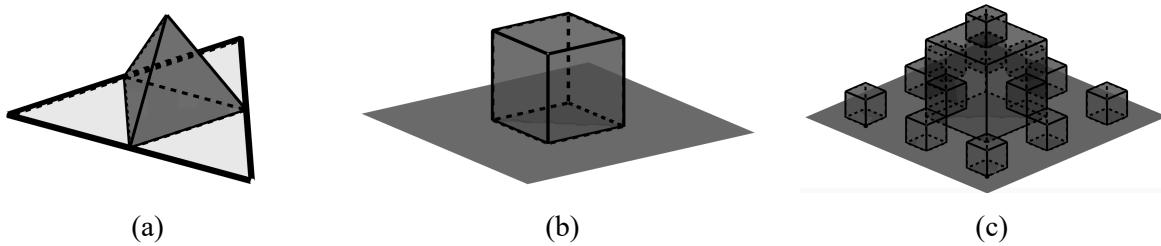
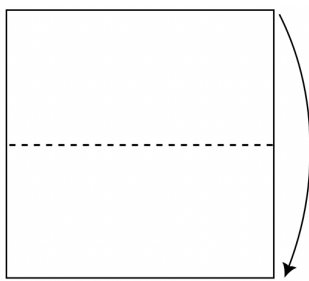
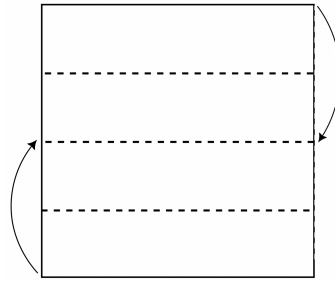


Figure 3: a) *Koch Surface*, iteration 1, b) *Square Koch Surface*, iteration 1, c) *Square Koch Surface*, iteration 2.

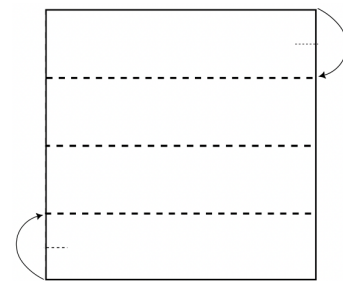
Origami Construction Steps



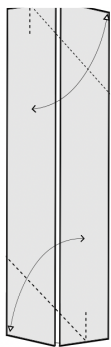
1. Start with one sheet of paper. Fold it in half, then unfold.



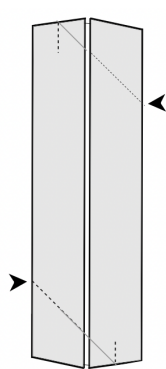
2. Fold both edges into the middle fold, then unfold.



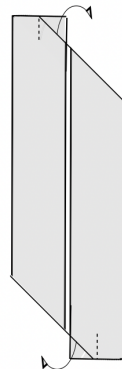
3. Pinch an eighth fold on each side, unfold, then refold step 2.



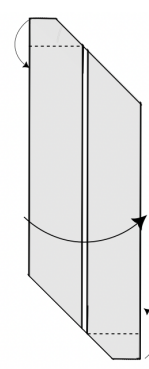
4. Fold corners, then unfold.



5. Reverse fold dotted line. Repeat on both sides.

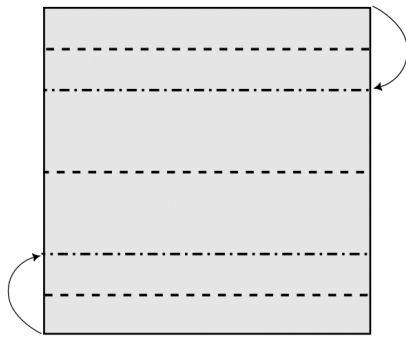


6. Tuck outer fold in at the top and bottom.

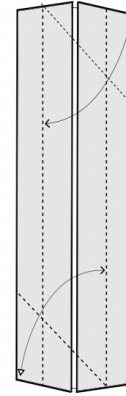
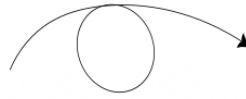


7. Fold in the ends of the model, so it is perpendicular to the midline. Repeat on both sides, then fold along the midline.

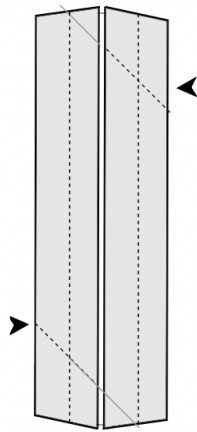
Figure 4: *Instructions for Unit One Modules.*



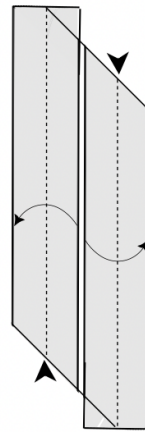
1. Begin by folding alternating fourths, starting with the color side up. Fold both edges into the fourth folds, unfold, then flip the paper over and refold Step 2 of Unit One Module Instructions.



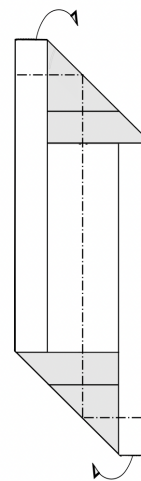
2. Fold corners, then unfold.



3. Reverse fold dotted line. Repeat on both sides.



4. Fold the middle to the edge, then collapse at the top of the unit at a 90 degree angle.



5. Fold the ends of the model back, so it is perpendicular to the midline. Repeat on other side.

Figure 5: Instructions for Unit Two Modules.

This model is constructed using five modules (Figure 6). Figures 6(a), 6(b), and 6(c) are *Unit One* modules (Figure 4), referred to as *Unit A* that use 15 x 15 cm paper, *Unit B* that use 5 x 5 cm paper, and *Unit C* that use 5 x 9 cm paper respectively.

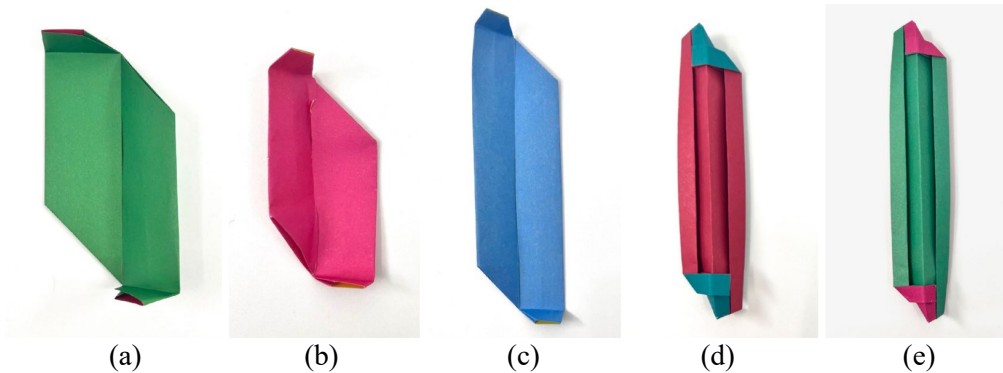


Figure 6: a) *Unit A*, b) *Unit B*, c) *Unit C*, d) *Unit D*, e) *Unit E*.

Figures 6(d) and 6(e) are *Unit Two* modules (Figure 5) that use 5 x 10 cm paper and 5 x 13 cm paper, respectively. They are labeled as *Unit D* (Figure 6(d)) and *Unit E* (Figure 6(e)), respectively. The model with a size of 36.8 cm x 36.8 cm x 24.4 cm requires 120 green Unit A modules, 200 red and 296 multicolored Unit B modules, 100 red and 88 multicolored Unit C modules, 60 multicolored Unit D modules, and 20 multicolored Unit E modules.

To start building, you will need four Unit D modules. Figure 7(a) shows the connection of two of these modules. Start by inserting the flap of one Unit D module into the open pocket of the other. Carefully wrap the flap of the second unit into the pocket of the first module. Once you have connected four of these modules, you will have created a D Square (as shown in Figure 7(b)). E Rectangles are rectangles that require two Unit D pieces and two Unit E modules (Figure 7(c)).

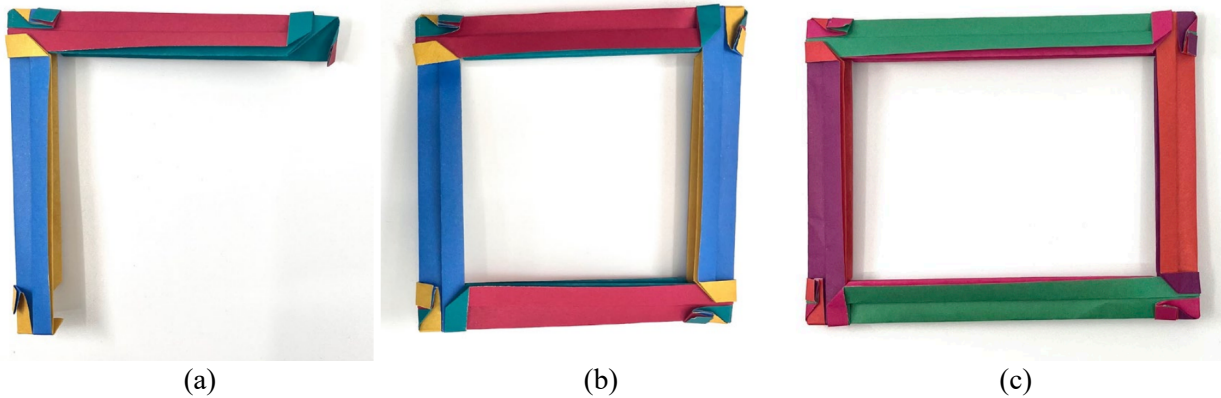


Figure 7: a) Two Unit D pieces connected, b) Four Unit D pieces creating a D Square, c) Two Unit D and two Unit E pieces creating an E Rectangle.

To build a cube, take twelve Unit B modules (Figure 6(b)) and connect them. Figure 8(a) shows how to connect two of these modules. Figure 8(b) provides a visual representation of the whole cube. The color of the cubes is insignificant. You will need to construct a total of ten cubes.

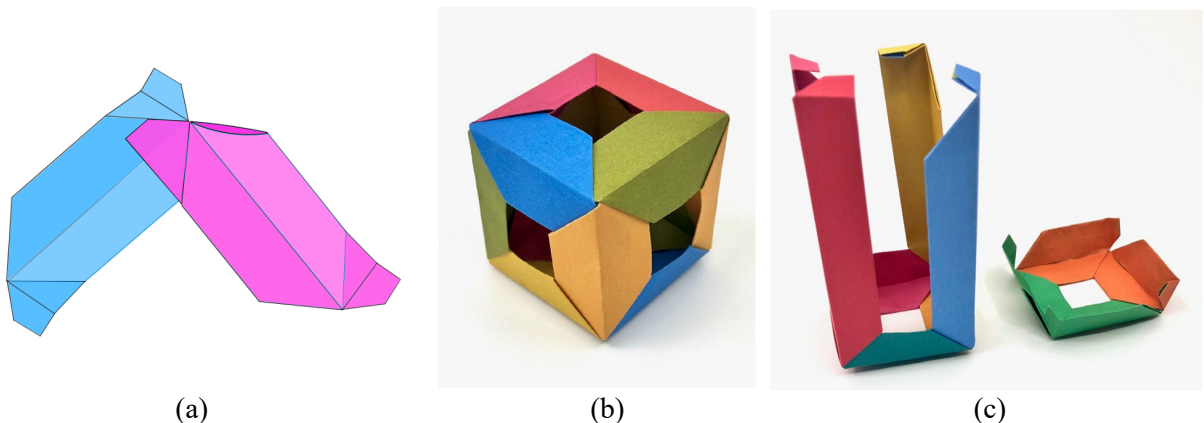


Figure 8: a) A brief diagram of the process of connecting two Unit One modules, b) A finished cube, c) An unfinished rectangular prism, which will be completed later in the building process.

To construct parts of rectangular prisms, use both Unit B and C modules, as shown in Figure 8(c). This process is similar to building cubes with the only difference being that the top of the rectangular prism will not be attached. It is important to note that the color of some of these prisms is significant. You need to

build a total of 47 prisms, with 25 of them being mono-colored (red is used in this example), and the remaining 22 can be of any color.

To begin the construction process, gather an unfinished prism (Figure 8(c)), two D Squares (Figure 7(b)), and one E Rectangle (Figure 7(c)). Begin by placing one D Square and E Rectangle flat inside the prism. Be sure to place the shorter end of the E Rectangle, as seen in Figure 9(a). Next, vertically put the second D square within both the prism and the long edge of the E Rectangle. Finally, carefully complete the prism to lock all the units.

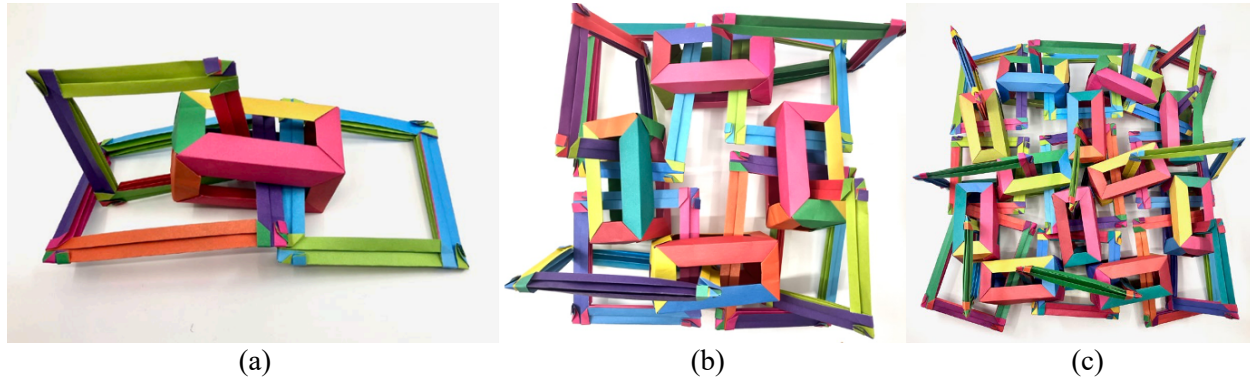


Figure 9: *a) Connection of two D Squares and an E Rectangle piece, b) Connection of four prisms with D Squares and E Rectangles, c) Connection of all D Squares and E Rectangles.*

To complete this step, extend this process to create a 3 x 3 grid of D Squares and E Rectangles (Figure 9(c)). Figure 9(b) shows the model while creating the 3 x 3 grid. To avoid D Squares and E Rectangles from overlapping, alternate between placing E Rectangles flat and D Squares vertically, and vice versa. Note that the prisms that connect D Squares and E Rectangles can be multicolored. As later seen in Figure 10(b), a checkerboard-like pattern of D Square and E Rectangle modules is used for both aesthetic purposes and to create more space for future steps.

Add the red prisms to the edges of the outer D Squares and E Rectangles (Figure 10(a)). Once a D Square or E Rectangle has four prisms surrounding it, make sure to position the prisms and insert one of the small cubes that were made previously (Figure 10(b)).

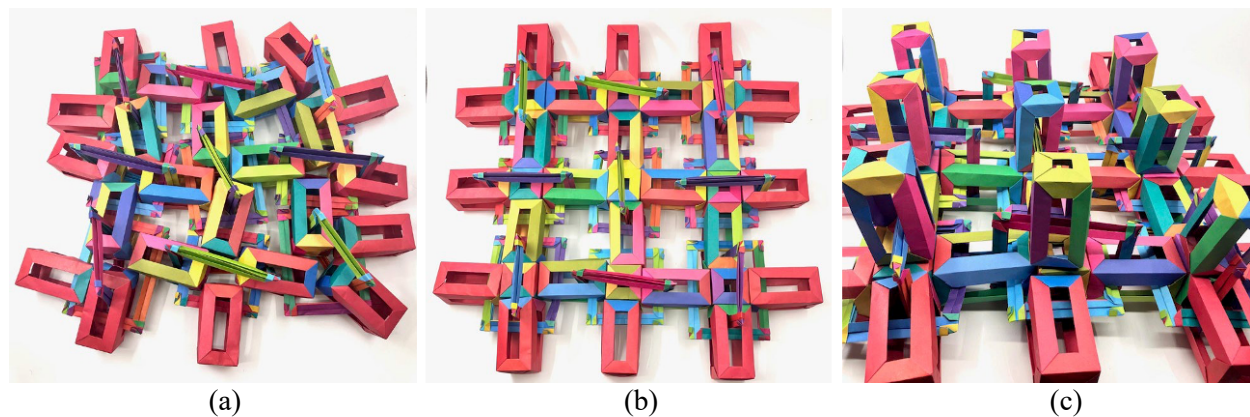


Figure 10: *a) Addition of mono-colored prisms to the edges of the model, b) Addition of cubes inserted between four prisms, c) Addition of prisms on the faces of revealed cubes.*

Continue by connecting nine multicolored prisms to the revealed faces of the inserted cubes (Figure 10(c)). Flip the model over, and attach a multicolored prism over the face of the middle cube, along with five mono-colored prisms, a cube, and the remaining D Squares and E Rectangles (Figure 11(a)). Finish

the frame of the model by attaching eight mono-colored prisms to the surrounding faces of the cubes (Figure 11(b)).

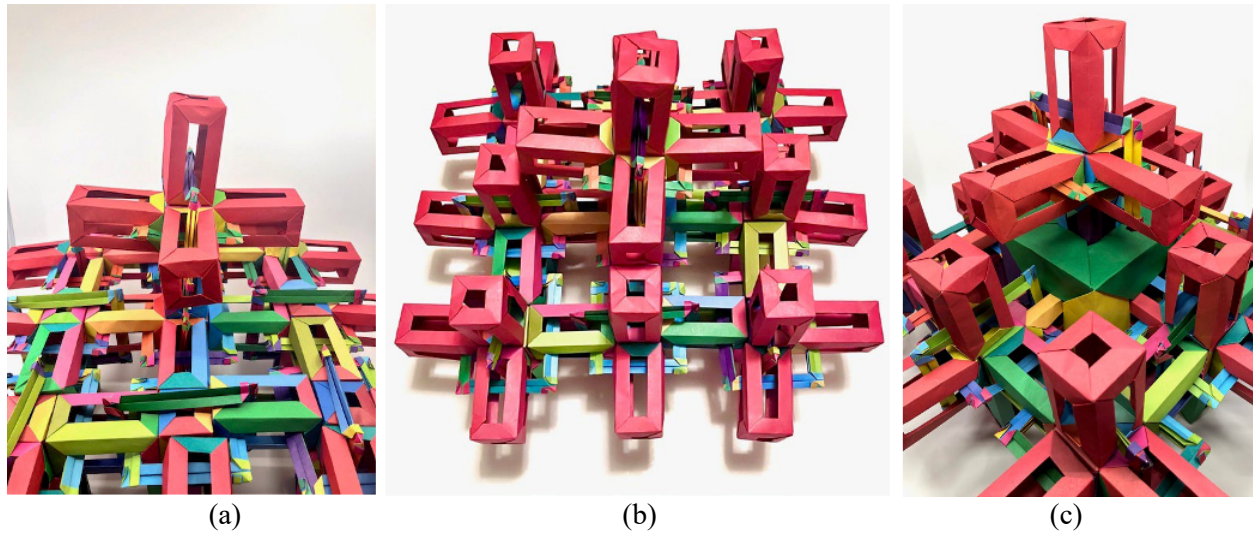


Figure 11: *a) Addition of another D Square and E Rectangle connection, with mono-colored prisms attached to each face, b) Addition of mono-colored prisms to surrounding faces of cubes, c) Construction of a cube around a section.*

Using 120 Unit A modules, begin constructing cubes around each section of prisms, and D Square and E Rectangle modules (Figure 11(c)). Note that the connections of Unit A modules are the same as the prisms and cubes used in this model (Figure 8(b)). Repeating this process around all ten sections of the model results in Figure 1, which shows a completed Square Koch Surface.

Constructing the Full Cube

Constructing a cube out of these pieces is a long but worthwhile process. This Square Koch Surface represents the first step of this process. By treating this model as the face of a cube, extending the model in the form of a cube is the next step to this fractal. The beauty of Square Koch Surfaces and Koch Cubes is their recursive identity. This process can be replicated, and a third iteration Square Koch Surface and Square Koch Cube is possible.

Acknowledgements

Thank you to Rona Gurkewitz for giving feedback on the instructions and diagrams. Thank you to Eve Torrence for inspiring and mentoring me to complete this paper. Thank you to Elisabeth Heathfield for helping me prepare this workshop. The diagrams were created using the Inkscape software application [3]. Figures 2 and 3 were made using Geogebra [2].

References

- [1] R. Dickau. Wolfram. *Square Koch Fractal Surface*. <https://demonstrations.wolfram.com/SquareKochFractalSurface/>
- [2] *Geogebra*. <https://www.geogebra.org/>
- [3] *Inkscape*. <https://inkscape.org/>