Dyeing to Make an Orbifold

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Abstract

Workshop participants will learn why only seven of the orbifolds corresponding to the seventeen plane symmetry groups can be folded and dyed from an uncut piece of paper. This understanding will be utilized in dyeing cotton bandanas with one of the seven available symmetry patterns using the technique of itajime shibori.

Introduction

This workshop will allow participants to have the aha experience of connecting the folding and decorating of an orbifold with the subsequent design produced on the flattened surface through clamp and resist (itajime shibori) dyeing. The purpose of this manuscript is to create a source in the literature where people wishing to shibori dye their own plane wallpaper patterns can learn to do so. Furthermore, the paper will enable individuals to incorporate the acitivity of shibori dyeing cloth into their mathematical arts programs. The purpose of the activity is to help participants to better understand the relationship between wallpaper patterns and oribfolds.

Bridges participants have benefitted from two phenomenal workshops related to constructing orbifolds: Hart (2013) [5] Orbifold and Cut and Gould & Gould (2020) [6] Bringing Orbifolds out of the Plane: Kaleidoscopes, Gyrations, Wonders, and Miracles. The distinction between those workshops and this one is the focus on the medium of itajime shibori cloth dyeing. We note that all of these workshops were informed by the book The Symmetries of Things [2], which is a tremendous resource for learning about orbifolds.

Ideas from this workshop can be flexibly used with people ranging from young kids to research mathematicians. Rather than a full workshop in which the goal is to develop deep mathematical understanding through tactile experience and reflection, what we describe could be implemented as a drop-in activity taking a few minutes. Needless to say, we expect people to learn more the longer and more deeply they engage with the ideas and play with the possibilities. With all of that in mind, we present an ambitious set of student learning objectives.

Student Learning Objectives:

1. Appreciate the mathematics behind the folding diagrams presented. (For example, understand why the folding diagrams yield the correct angles on the triangles, understand why the resulting dyed pieces have only points with mirror symmetries, and not rotational symmetries, and understand why the folding diagrams yield kaliedscopic points with the orders that they do.)
2. Gain a deep understanding of the connection between the physical creation of the orbifold from the piece of cloth, the constraints of the itajime process, and the reason only seven of the seventeen wallpaper patterns can be realized with itajime shibori.
3. Learn the Conway-Thurston notation for some of the orbifolds.
4. Understand the importance of clearly identifying and stating physical constraints when trying to mathematically characterize the fiber arts.

Following an introduction to the art of itajime shibori dyeing, described in Section , Yackel set out to dye vast number of fabric pieces, hoping in the process to exhibit all seventeen plane symmetry types. As is often the case, the reality of the process created a surprise! Only seven of the seventeen types are possible using itajime shibori. The interaction between the mathematics and the fiber art is thoroughly explained in the paper [12]; it relies on the theory of orbifolds. In the workshop, we only intend to discuss orbifolds for plane symmetries without delving into their algebraic/geometric constructions. People wishing to use the workshop instructions below for individuals not interested in orbifolds can absolutely do so.

To complete the monumental task of dyeing the many fabric items, the authors worked together, allowing observation of more and less successful techniques. In this workshop, we give participants an opportunity to create their own dyed itajime shibori fabric pieces for the purposes of having an embodied understanding of the relationship between the orbifolds and the resulting patterns. Makers of all stripes will attest to the qualitative difference in understanding that results creating a piece of (mathematical) artwork as opposed to simply thinking about the mathematics itself.

Background

The Japanese art of shibori fabric dyeing, of which tie-dye is a subgenre, has been experiencing a resurgence in popularity in the West, as noted in [8]. Readers interested in the pure Japanese version of the craft may consult [10]. For the purposes of this article, we will consider the westernized version of shibori, which uses compression to create what is called a *resist*, or an area where dye will not penetrate. This contrasts with other dye techniques that use different methods to create areas of resist; for example, batik uses wax and, according to Becker [1], adire eleko, originating from Nigeria, uses cassava based paste. Different types of shibori employ various method for creating compression. Nui shibori uses basting stitches about which fabric is subsequently tightly gathered [4], shown in Figure 1a. Kanoko shibori involves tightly binding the base of a regions of fabric with thread, which creates a simple closed curve or a white blob, as shown in Figure 1b. The topic of this paper and workshop is itajime shibori, in which the fabric is only folded and then the cloth is clamped between a matched pair of resist blocks, shown in Figure 1c.

(a) *Nui* **(b)** *Kanoko* **(c)** *Itajime*

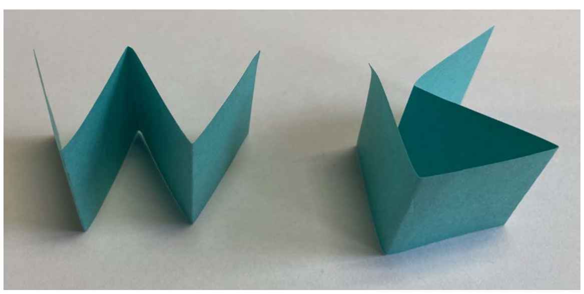
Figure 1: *The results of using various shibori resist techniques. Shibori images used with permission from Jessica Kauffman [4].*

The striking geometric results of itajime shibori call to mind *wallpaper patterns*, which are defined as patterns that repeat infinitely in two linearly independent directions. While any given piece of cloth is finite, we can imagine that piece of cloth extending indefinitely to cover the plane. Likewise on that cloth, we can extend patterns having repeats in two independent directions to plane patterns. In imagining how that pattern could have come to be created on the cloth, it is helpful to think about the following notions. The plane symmetry group associated to a given wallpaper pattern is the set of isometries of the pattern under the group action of composition. Each plane symmetry group is associated to what Thurston dubbed an orbifold. [11] The *orbifold* of a plane symmetry group corresponds to the geometric object obtained by identifying points in the plane that share an orbit under the group action. That is, we need to fold the fabric into its orbifold, then clamp on resist blocks and dye.

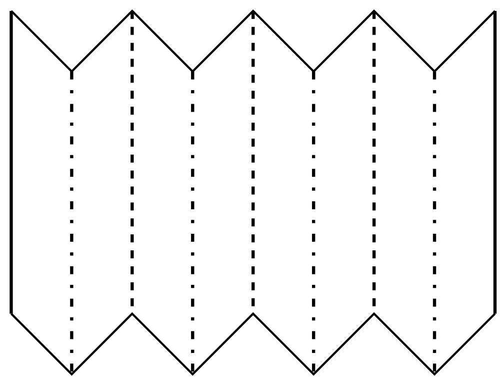
Avid students of symmetries will know that there are seventeen plane symmetry groups, and we encourage others to consult their favorite resource, such as [7, p. 40-42] or [9]. In a previous paper, Yackel has laid out a mathematical argument for why only seven of the seventeen plane symmetry groups are achievable via itajime shibori [12]. The crux of the argument rests on two things. First, because the cloth is only folded, but never cut, no achievable orbifold may have a cone point, which would result from a point of pure rotational symmetry. Second, all orbifolds must be achievable in three-dimensional space.

Folding

The mathematical portion of the work of itajime shibori dyeing lies in understanding how to fold the cloth into the relevant orbifolds. In this section orbifold folding directions along with practical tips for making the final dyed pieces come out as nicely as possible. Crisp, clean folds most accurately represent the intended wallpaper pattern. For this reason, when we fold, we will make accordion folds whenever possible. That is, we will fold back and forth rather than trapping excess fabric in the interior of folds, as when we fold in half and then in half again. Compare the two folding types in Figure 2. Using too many layers of cloth typically



(a) *The left piece of paper was accordion folded, whereas the right piece of paper was not.*



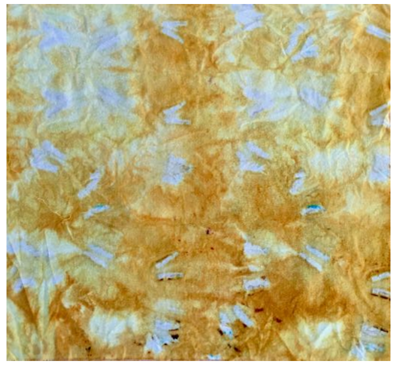
(b) *Folding diagram: the dashed lines are valley folds and the dot-dashed lines are mountain folds.*

Figure 2: *Visual directions for accordion folding.*

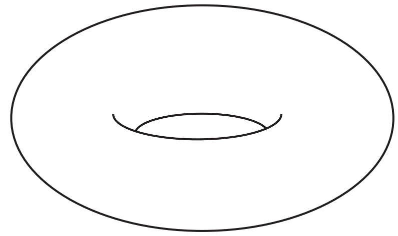
introduces error into the folding, which means that the wallpaper pattern will not be faithfully repeated. In addition, the many layers prevent dye from fully penetrating around the resists. The difference in penetration alters the look of the pattern as it is repeated. This is easily visible in Figure 9(c), where one half of each repeat shows the washers used for the resist, and the other half has only negative space blanks in the corresponding spots.

The order in which we present the following orbifolds with their folding diagrams relates to the mathematical structure, not necessarily to their magnificence when produced in shibori. Figure 3 shows how to fold the orbifold for the torus, which involves folding the cloth in the same direction a number of times, as if rolling it. Then we turn the cloth in a perpendicular direction, and we fold down the cloth, a number of times, as if rolling up sleeves of a garment. This creates the torus shape. Unfortunately, creating crisp folds is nearly impossible if using a woven cloth without elasticity. This is made even more difficult when the cloth is wet. Because the resists are placed after the orbifold is created, mirrored pairs of resist shapes will occur in this shibori dyed piece.

(a) *Roll the cloth to create a double covered cylinder. Then fold the cloth down, like sleeves, three times to create a torus wrapped twice through the hole.*



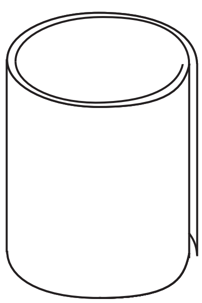
(b) *The torus orbifold yields a difficult to discern pattern created only by translations.*



(c) *Torus orbifold.*

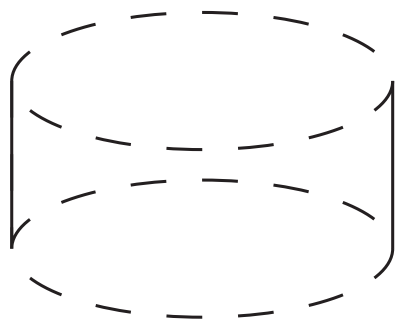
Figure 3: *The torus orbifold yields the o pattern in Conway notation (p1 in IUC notation).*

In order to make all of the remaining orbifolds, begin by folding the cloth into a long rectangle using accordion folds. The goal is to have sufficient repeats to observe the pattern, but to avoid having too many cloth layers. We suggest beginning with an approximately square cloth and accordion folding into fourths. The folding steps given in the diagrams follow after this initial step. In order to make the mirrored annulus we wrap the long accordion folded cloth into a cylinder shape, making sure the cloth wraps around the shape at least two full turns so that the wallpaper pattern has at least two repetitions in the direction corresponding to the wrapping direction that traverses the cylinder. Figures 4, 5, 6, 7, 8, and 9 each consists of a triptych containing a folding diagram (left), the corresponding orbifold diagram (right), and a photo of a corresponding shibori dyed cloth (center).



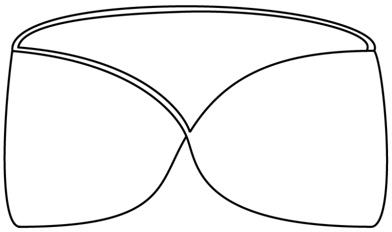
(a) *Roll the (accordion folded) cloth into a double cover of a cylinder.*

(b) *The mirrored annulus orbifold gives a pattern with two types of vertical mirror lines and a vertical shift.*

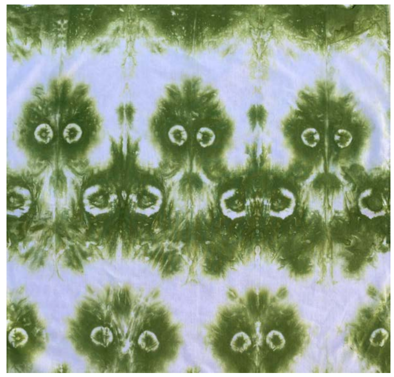


(c) *Mirrored annulus orbifold. Dotted lines indicate mirrors.*

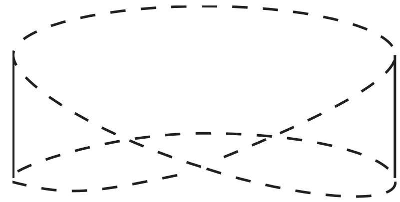
Figure 4: *The mirrored annulus orbifold corresponds to the $**$ pattern in Conway notation (pm in IUC notation).*



(a) *Make a perpendicular fold down the center of the (accordion folded) cloth. On one side of the fold create a Möbius band. Follow the twist of the Möbius band with the other half of the fabric strip to make a double cover.*

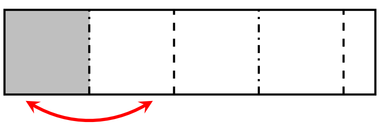


(b) *The Möbius band orbifold features glides and mirrors in the dyed product.*

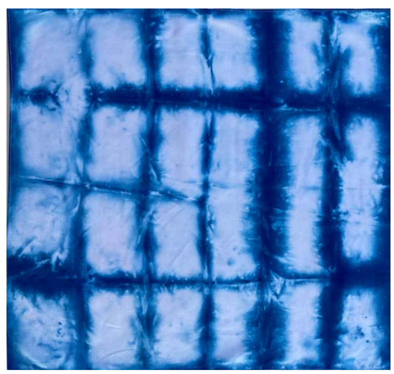


(c) *Mirrored Möbius orbifold. Dotted lines indicate mirrors.*

Figure 5: *The mirrored Möbius orbifold results in the* $*\times$ *pattern in Conway notation (cm in IUC notation).*



(a) *Accordion fold (the accordion folded) cloth in the perpendicular direction.*



(b) *The mirrored rectangle has perpendicular mirror lines.*

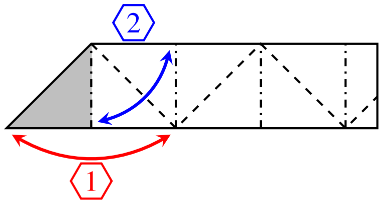
(c) *Rectangular orbifold with mirrored edges.*

Figure 6: *The rectangular orbifold with mirrored edges yields wallpaper pattern* $*2222$ *in Conway notation (pmm in IUC notation).*

Dyeing

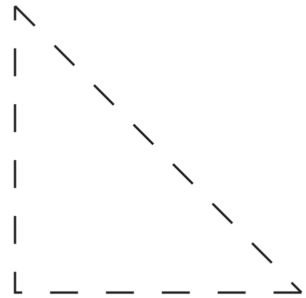
We have used both Procion and Rit dyes. Procion dyes require fabric to be presoaked in a soda ash solution and for dyes to set for twenty-four hours before rinsing. Soda ash soaked fabric is harsh on hands so rubber gloves are a necessity, even while folding. In addition, precisely folding wet cloth is much more difficult than precisely folding dry cloth, for which one can use an iron! However, dry cloth wicks moisture, so clamped dry cloth will likely wick liquid into the clamped region in an uncontrolled fashion. On the other hand, clamping dry folded fabric, then soaking in soda ash or water, and then dyeing appears to give a more precise dye outcome. In a workshop setting, Rit dyes may be favorable, because they require a shorter set time of only thirty minutes versus an entire day. Excess Rit dye is absorbed into the white space, making the pattern less stark.

For clamps and resist blocks, binder clips combined with coins and washers create nice designs. Laser-cut acrylic pieces are especially effective as resists. We conjecture this is a result of the evenly distributed pressure by having a very smooth, flat surface. Spring-style workshop clamps are quick to use. These also allow for compression farther away from the edge of the cloth. Rubber bands give an imprecise compression



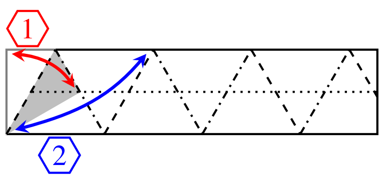
(a) *Fold (the accordion folded cloth) in a 45° angle a the vertex. Step 1: Fold backwards over the vertical line. Step 2: Fold forwards over the diagonal. Step 3: Repeat steps 1 and 2.*

(b) *The mirrored right triangle orbifold makes two types of four-fold kaleidoscopic points and one type of two-fold kaleidoscopic point.*



(c) *Right triangle orbifold with mirrored edges.*

Figure 7: *The right triangle orbifold with mirrored edges results in the wallpaper pattern* ∗442 *in Conway notation (p4m in IUC notation).*



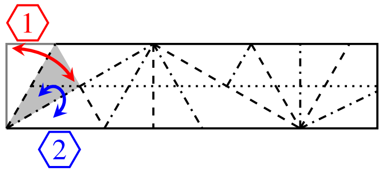
(a) *Step 1: Slightly crease the (accordion folded) cloth lengthwise to identify the middle. Pivot about one corner vertex while folding forward to bring the opposite corner to the middle crease, creating a* 60° *angle. Step 2: Fold backwards matching the angle created in Step 1. Step 3: Fold forwards matching the angle created in Step 2. Repeat steps 2 and 3.*

(b) *The mirrored equilateral triangle orbifold results in a pattern with three types of three-fold kaleidoscopic points.*

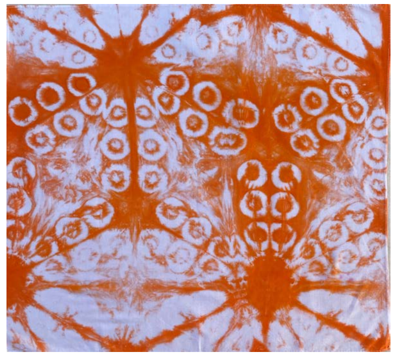
(c) *Mirrored equilateral triangle orbifold.*

Figure 8: *The mirrored equilateral triangle orbifold corresponds to the wallpaper pattern* ∗333 *in Conway notation (p3m1 in IUC notation).*

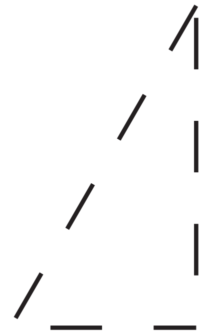
location, but otherwise are effective when they don't break, and they provide an inexpensive alternative. At times, paired tongue depressors create a nice linear resist. Other times it is difficult to obtain a tight enough bind to prevent dye from penetrating the cloth. All of these make-shift items stand in for traditional paired resist shapes bound with twine.



(a) *Step 1: Fold as for ∗333 in 8 above. Step 2: Mountain fold. Further steps: Alternate the valley and mountain folds shown, working from left to right. The final result will be a 30-60-90 triangle.*



(b) *The mirrored 30-60-90 right triangle orbifold wallpaper pattern has one type of two-fold, one type of three-fold and one type of six-fold kaleidscopic point.*



(c) *The mirrored 30-60-90 right triangle orbifold.*

Figure 9: *The mirrored 30-60-90 right triangle orbifold gives the wallpaper pattern ∗632 in Conway notation (p6m in IUC notation).*

Workshop

For this workshop, we have compiled a materials list and laid out a tentative structure.

Materials:

- Square Bandanas that have been scoured in Synthrapol, so that they are ready to absorb dye.
- Rubber gloves (for protecting hands)
- 2 plastic drop cloths–one for table, one for floor under table
- Dye (Rit or procion)
- Pitcher, electric kettle (for water)
- Tongs, junk spoon (to stir dye vat)
- 2 Buckets: one for dyeing, one as a holding place for wet work before and after the dye bath.
- Clamps (Need plenty, including ones with long mouth. Can use binder clips, clothes pins, and workshop clamps. These must be tight.)
- Resist block pairs (quarters, half-dollars, or other coins; metal washers; laser-cut shapes; thin plywood shapes, such as triangles, squares, or rectangles; tongue depressors; etc.), rubber bands
- Thread, needle, masking tape, sharpie (for labelling work while it's in the dye bath)
- Zip lock sandwich bags (for people to take home bandana)
- (optional) Color catching dryer sheet to give out for first wash at home

Structure:

Part 1 (20-30 minutes): The workshop will begin with a brief discussion of itajime shibori dyeing, including a slideshow of images. We will go on to identify pattern types we observe in the images and wonder about how those could be obtained by the dyers. We will then discuss wallpaper patterns and their relationships to orbifolds. Participants will be given an array of pre-dyed handkerchiefs to practice folding into their dyed shapes.

Part 2 (20-30 minutes): After each person chooses the wallpaper pattern they wish to emulate, they will be given the opportunity to fold a handkerchief into that orbifold, clamp it, and give it to one of the leaders to dye.

Part 3 (20-30 minutes): While the pieces dye and after they emerge from the dye bath, the participants will examine pre-dyed and just dyed pieces to consider the deep relationship between the folding, the placement of the resists, and the patterns resulting in the dyed pieces.

Conclusion

The orbifold notation utilized in this paper describes the symmetry groups that correspond to the seventeen wallpaper patterns. Without cutting or sewing, the fabric handkerchief is folded into the orbifolds that allow the dyed pattern to repeat in a manner that gives the creator seven of the wallpaper patterns. The dye penetrates the outer layers of the folded domain more intensely than that of the inner layers. The variation of dyes between layers, imperfectness of the folded fabric, and the intensity of the dye lead to unique pieces. There is a deep connection between geometry, topology, and symmetry groups explored by folding and dyeing the fabric. The folding for each rectangular piece of fabric directly relates to the symmetry group being represented.

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