From Knot Diagrams to Crocheted Topological Surfaces

Shiying Dong

Greenwich, Connecticut, USA; shiyingdong@gmail.com

Abstract

This paper explains how to use knot or link diagrams to make topological crochet sculptures. Two methods are discussed. The main crochet project is the saddle trefoil surface.

Introduction

In my 2023 Bridges workshop on topological crochet [3] we learned mathematical motivation, some basic concepts, and crochet techniques for making seamless surface sculptures that highlight the underlying topological structure. The central process of topological crochet is to set up a *ribbon graph* using foundation chains and crochet rounds along the boundary of such a graph to achieve the desired size and form. This paper investigates the designing stage along with examples.

There are many points from which one can start a topological crochet sculpture; one such starting point is a knot or link diagram. Given such a diagram, numerous ways exist to design a surface bounded by the knot or link, such as the classic Seifert's algorithm. We will introduce two other ways: the *two-color algorithm* and the *saddle method*. The key idea underlying all these algorithms is to think of a knot or link diagram as part of a projection of a surface in 3D to 2D, such that the boundary of the surface is projected onto the knot or link in the diagram. The projection is almost certainly not 1-1, except in highly trivial cases. This leaves many ways to "restore" the surface, given its boundary image in 2D.

Together with [3], this paper has enough content to expand into a mini-series of four workshops, with each method having two workshops. One-time workshops can also focus on one method, either the designing stage or the example execution. For the Bridges 2024 workshop, we will focus on how to make the example for the saddle method, with an introductory sketching of the designing stage for about 10 minutes. The project is a *saddle trefoil* [6] [2] surface bounded by a (3,4)-torus knot. The participants will start from a pre-made foundation chain graph and continue with round 1. The goal of the workshop is to finish round 1. The participants can finish the rest after the workshop without too much trouble.

In [3], I included pictures of models designed with both methods. The following are a few more sculptures created using the saddle method.

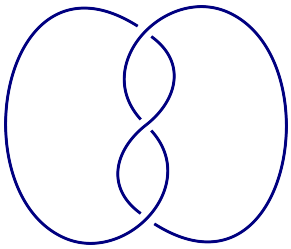
Figure 1: *Examples that are made using the saddle method.*

Two-Color Algorithm

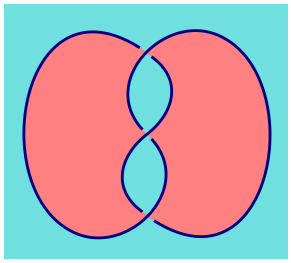
The classic Seifert's algorithm only produces orientable surfaces, and it can be hard to draw by hand when a knot or link diagram is complicated. Software such as [8] is available to help visualize the Seifert surfaces. However, many interesting surfaces are nonorientable. A simpler algorithm [1] exists to give rise to two possibly nonorientable surfaces from a given diagram. Let's call it the *two-color* algorithm. Following are its steps adapted to topological crochet:

1. Draw the knot or link diagram.
2. Color the whole paper with two colors, with the only constraint that no neighbor regions have the same color. The outer area is also assigned a color. We call this the *checkerboard* rule.
3. Pick one color and collect all the regions with that color, each representing a patch in the surface, which implies a chain stitch. We call this step *weeding*.
4. For each crossing in the original diagram, attach a twisted band joining the patches—the final surface results from all patches with all bands joining them. In topological crochet terms, this means a twisted foundation chain connecting the two corresponding chain stitches.

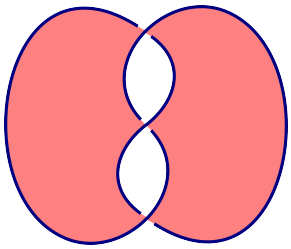
Let's see how it works from an example in Figure 2. Figure 2(a) is a knot diagram for a right-handed trefoil knot. Figures 2(b) and 2(c) are the coloring and weeding steps. Figure 2(d) shows two stitches joined by three bands, all twisted 180-degree in the same direction. Figure 2(e) is the drawing showing how to build the graph with one foundation chain. Figure 2(f) is what it looks like in crochet. Details of how to finish this project can be found in [3] and [4]. The reader can check that had we chosen the other color in Figure 2(b), then the final project would be a Möbius strip with three 180-degree twists. In this example, the same knot diagram gives rise to one orientable and one nonorientable surface.



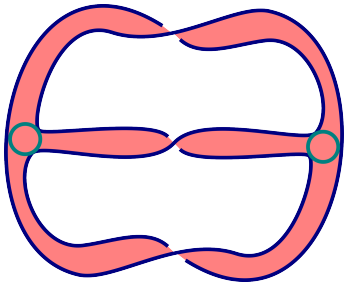
(a) *Knot diagram of a right-handed trefoil knot.*



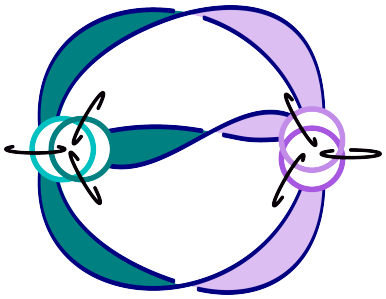
(b) *Colored using the checkerboard rule.*



(c) *Weed out one color and keep another.*



(d) *Deform the surface into a ribbon-like shape.*



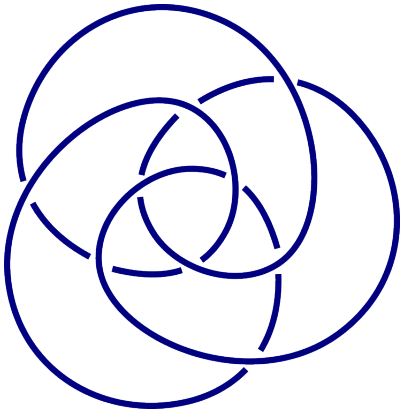
(e) *Foundation chain ribbon graph with markers.*

(f) *Finished foundation chain graph in practice.*

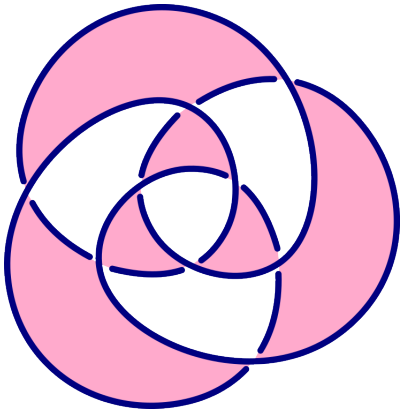
Figure 2: *Application of two-color algorithm on the trefoil knot.*

Saddle Method

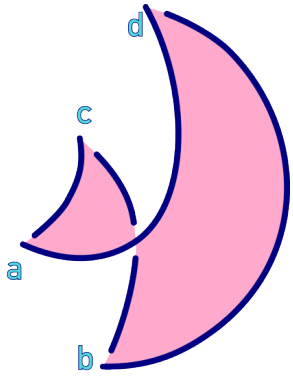
The saddle method could be considered an extension and variation of the two-color algorithm. When applicable, it can create more complex models than the two-color algorithm alone. It can be applied to knot or link diagram colorings that contain pairs of triangular regions joined by a crossing. An example of such a diagram is the (3,4)-torus knot shown in Figure 3(a) and its coloring in Figure 3(b). The shape comprises three copies of Figure 3(c) meeting at the open ends. Unlike the patches used in the two-color algorithm, this shape contains a crossing. Let's call it a *saddle patch*. Indeed, this shape is a projection of the neighborhood of a saddle point [7], which implies two directions go one way and two others go the opposite way. These two sets of directions are marked by *a, b* and *c, d*.



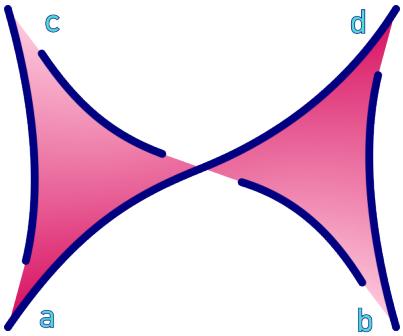
(a) *Knot diagram of a (3,4)-torus knot.*



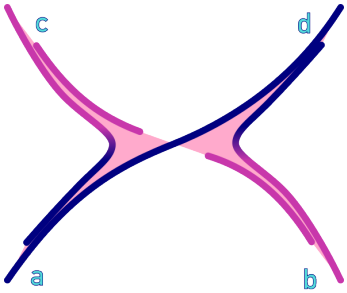
(b) *After coloring with two colors and weeding out one.*



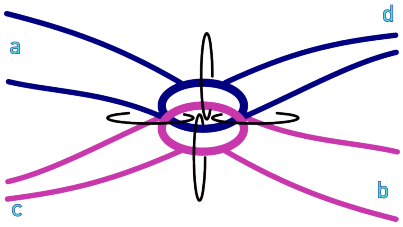
(c) *A single saddle patch with four open ends.*



(d) *A more symmetric drawing of the saddle patch.*



(e) *A skinny symmetric saddle patch looks like a knot crossing.*



(f) *Side view of the saddle point chain setup with stitch markers.*

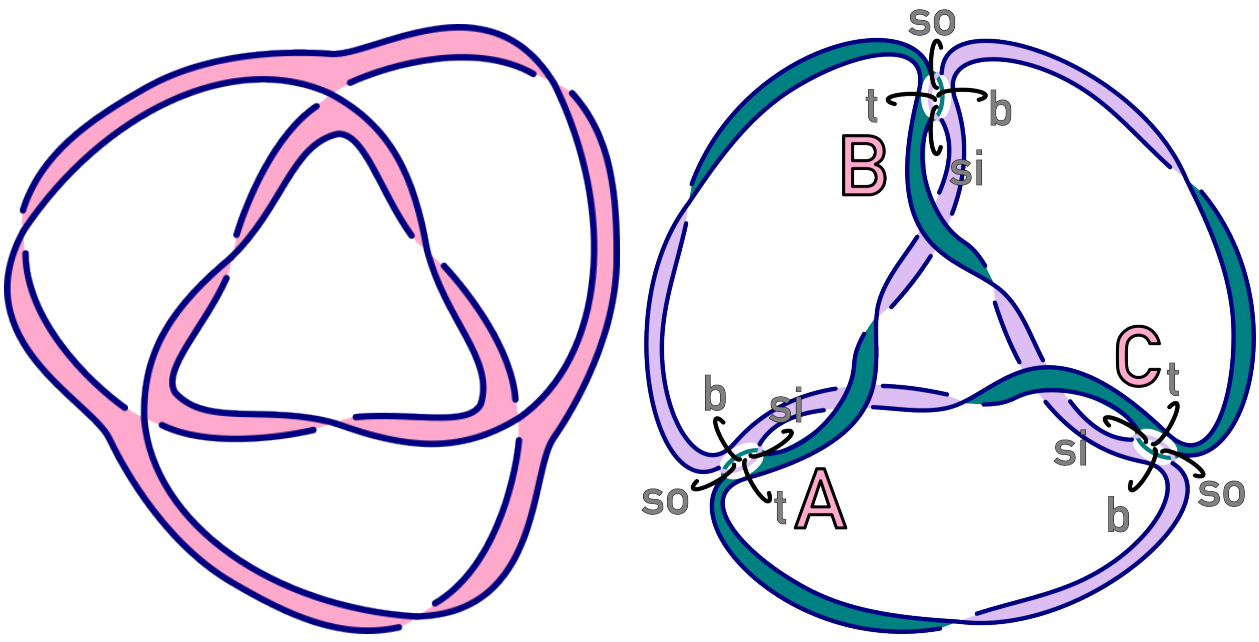
Figure 3: *Studying saddle points in a surface bounded by the (3,4)-torus knot.*

Converting a saddle patch into a foundation chain in crochet is quite different from what we did in the two-color algorithm. In Figure 3(d), we drew the saddle patch in a more symmetric fashion, and in Figure 3(e), we made the shape skinny to help visualize the foundation chains, and make it look like a knot crossing made from two bands. The top band (blue) goes southwest to northeast, and the bottom (pink) goes northwest to southeast. The middle stitches from these two bands will be identified as one at the saddle point. Moreover, the foundation chains are no longer "lying flat" but standing up. This causes difficulty in drawing chain diagrams in an ordinary way where every chain stitch for every patch faces the reader. A side view of the vicinity of the saddle point in Figure 3(e) is shown in Figure 3(f), in which the camera shoots from the south of Figure 3(e), the blue band is above the pink band, and they join at the middle stitch. We use four stitch markers to hold this shape in place temporarily: one at the top, one at the bottom, and one on each side. This arrangement can be tricky to perform at the beginning since if two bands form a cross, one naturally prefers that each band go straight through the middle point instead of having a bend. It takes some practice to get used to this.

Not all knot diagram colorings can be decomposed into a set of saddle patches; most have saddle patch areas and something else. In that case, the saddle method can be applied to the saddle patch areas while the rest is modeled using the two-color algorithm. The resulting surface differs depending on which subset of the patches are modeled using the saddle method.

Saddle Trefoil

Figure 3(b) shows our main crochet project of the workshop: the saddle trefoil. It contains three saddle patches that form a toroid. Therefore, we can sketch the foundation chain graph in Figure 4(a). In Figure 4(b), we draw the graph in full detail, where the front side of the foundation chain is indicated in teal and the back in lavender. We label the saddle points with "A, B, C," and the stitch markers "t (top), b (bottom), si (side in), so (side out)." The top and bottom stitch markers at the saddle point A will be referred to as A-t and A-b, and the others are labeled similarly.



(a) *Chain graph of the saddle trefoil.* (b) *Full chain ribbon graph with labels.*

Figure 4: *Diagrams of the foundation chain ribbon graph of the saddle trefoil.*

The seeming discrepancy between Figures 4(a) and 4(b) is not a mistake in the drawing; it is because in Figures 4(a), the saddle points are standing up, while in Figures 4(b), we pushed each saddle point down so its details can be viewed from the top.

If we ignore the fact that stitches are joined at three saddle points, then the centerline of the foundation chain forms a trefoil knot, hence the name "saddle trefoil."

With three saddle patches and the trefoil knot in mind, we are ready to crochet. Materials needed for this project include two colors of worsted-weight yarn (one for the body and one for the boundary) and 54 inches of 250 lb *monofilament fishing line* with *non-insulated butt connectors* of type 18-22 AWG. We will use a size 7 (4.5mm) crochet hook and a few split ring stitch markers of various colors. Scissors, darning needles, wire cutters, and pliers can be shared among participants. The steps are described as follows and illustrated in Figures 5–6.

1. Build the foundation chain ribbon graph. Make a foundation chain of length $9m + 1$ for some chosen m. In Figure 5(a), $m = 12$. Another good choice is $m = 8$, for which step 3 is skipped. Mark stitches $m + 1, 4m + 1, 7m + 1$ with stitch markers (white in Figure 5(a)). Then mark stitches $2m + 1, 5m + 1, 8m + 1$ with stitch markers, ideally with a second color (blue in Figure 5(a)). One can also mark the end stitches (pink in Figure 5(a)), but it's not necessary. Tie a trefoil knot with the chain and make sure there is *no* twist. Arrange the knot gently so that stitches marked in one color (white in Figure 5(b)) are right on top of stitches marked in the other color (blue in Figure 5(b)). Join the end stitches by pulling the working yarn loop through the first stitch and ch 1. Start a few single crochet (sc) stitches so the join is solid. Put a stitch marker through the joined end stitches on the opposite side of the chain from where sc is performed (Figure 5(c)). Remove the crochet hook temporarily.

Very carefully, make the whole chain stand up while not introducing any local twist nor making changes to the shape of the knot (Figure 5(d)). The front side of chain faces out on the three trefoil knot lobes.

At each crossing, slide the chains down together so that the front sides face each other. Then insert two stitch markers on the side of joined stitches. Make sure that the stitch markers will go through both stitches that are marked. Ideally, the two side stitch markers are the same color (white in Figure 5(e)). It might be a good idea to finish this step for all three saddle points before moving on. Then, for each saddle point, use a stitch marker of the second color to go through both stitches at the bottom (Figure 5(f)). Finally, pick a stitch marker of the second color to go through the top of both stitches (Figure 5(g)). Make sure all four markers at a saddle point go through the same hole and don't hook through each other. Figure 5(h) is what it looks like after all the stitch markers are in place, which matches Figure 4(b).

2. Round 1: removing the stitch markers. Insert the crochet hook from where it was removed, and continue single crocheting (Figure 5(i)). Crochet round 1 with the following guideline: double crochet (dc) at the saddle points, half-double crochet (hdc) at the two neighbors of each saddle point, and sc the rest. After round 1, there will be four dc stitches around each saddle point, one for each stitch marker. Remove a stitch marker whenever you crochet past one.

Figure 5(j) shows how your work should look when first approaching a saddle point. It is essential to adjust your work to be *flat* around the saddle point. Figure 5(k) is after finishing the hdc right before the saddle point, and Figure 5(l) is after one stitch marker is removed and replaced by a dc. Figure 5(m) shows how it looks after we crochet a few stitches on the following chain segment. Figure 5(n) shows how your work should look when approaching a second saddle point. Everything around the saddle point is flat, and the top white stitch marker will be removed (Figure 5(o)). Figure 5(p) shows a saddle point neighborhood after removing two stitch markers. Remember that the setup of our foundation chain guarantees that round 1 will be a (3,4)-torus knot as long as we crochet along the boundary. To stay along the boundary, one needs to constantly hold the piece so that the small neighborhood around the stitch one is working on is laid out nicely, *flat* and *untwisted*. One needs to be very familiar with what an untwisted chain segment and a flat

(a) *Chain 109. Mark stitches 13, 49, and 85 with white and 25, 61, and 97 with blue.*

(b) *A loosely tied trefoil knot with white markers above the blue ones. Chain starts at left.*

(c) *Join the end stitches and start a few sc. Put a stitch marker on the other side.*

(d) *Make the foundation chain stand up without changing the trefoil knot.*

(e) *For each crossing, put two markers through both marked stitches from both sides.*

(f) *Put another marker through the two stitches from the bottom.*

(g) *Put a fourth marker through the two stitches from the top.*

(h) *Finished foundation chain graph that matches with Figure 4*(b).

(i) *Start crocheting round 1 with single crochet.*

(j) *Approaching the first saddle point.*

(k) *Hdc before the saddle point.*

(l) *Dc at the saddle point.*

(m) *Crochet along the second segment.*

(n) *Approach the second saddle point.*

(o) *Dc at the saddle point.*

(p) *A saddle point after two markers are removed.*

Figure 5: *Making the saddle trefoil.*

saddle point look like. Tracking the order in which each segment is visited might be beneficial in preventing mistakes. To this end, label the stitch markers in Figure 5(h) as in Figure 4(b). The starting point of this round is the top side of the middle of the lower segment. A right-handed crocheter will visit the stitch markers in the order of *(A-t, B-si, C-b, A-so, B-t, C-si, A-b, B-so, C-t, A-si, B-b, C-so)*. For a left-handed crocheter, it's the reverse. Notice this sequence is simply the product of two cyclic orders *(A, B, C)* and *(t, si, b, so)*. These two cyclic orders show the trace of the torus knot in the longitude and meridian directions.

Even though the initial trefoil knotted chain is untwisted and orientable with a front and back side, the final project's crocheted surface has only one side. As a result, one has to crochet sometimes from the front and sometimes from the back side of the foundation chain. The side switching happens when one crosses a saddle point and removes a *side* stitch marker.

Unlike the example in [3], where foundation chain segments get all crocheted on one side before the other side starts, in this project, some strands get crocheted on from both sides before other strands are crocheted on at all.

After all stitch markers are removed and replaced by dc, and we are back at the first stitch of this round, use a slip stitch to end the round (Figure 6(a)) if you are doing another round; otherwise, fasten off and cut the yarn. For those newer to crochet, dc and hdc may be replaced by sc for all stitches.

3. Round 2: keep growing. This step is optional. Ch 1 and sc all around. Fasten off when the round is finished, cut the yarn, and weave in the ends (Figure 6(b)). For a large m, more rounds can be added.

4. Round 3: color and wire the boundary. Cut the fishing line (or other material you choose) to the correct length and insert one end into the connector. Use the second color of yarn to sc around the wire and along the boundary, starting from any point (Figure 6(c)). Before finishing, trim the excess supportive material, insert the other end into the connector, then complete the round and fasten off. The finished model is shown in Figure 6(d).

(a) *End of round 1.* (b) *End of round 2.* (c) *Starting the boundary with the fishing line.* (d) *The final work.*

Figure 6: *Finishing the saddle trefoil.*

In the first two steps, we build (using stitch markers) and reinforce (removing the stitch markers) the foundation chain graph. In the third step (round 2), we simply make it wider. In the fourth step (round 3), we finish the work by decorating it with some desired colors and using a nylon line to support the whole surface. The third step can be omitted entirely for a smaller-size model.

The split ring stitch markers we used in this paper are convenient to put on and take off, but they can fall off during round 1. Locking stitch markers, short strands of yarn, or something else that makes it easy to hold the stitches temporarily can provide more secure lockings. The reader can try all the options to determine what works best for them. The fishing line works very well in this project. If unavailable, it can be replaced with other types of supportive wire, such as weed wacker lines or metal wire, as long as the gauge matches the strength of the yarn stitches.

It is important to remember the following when learning topological crochet:

- At step 1, a stitch marker must be used between *every* pair of neighboring edges in every vertex where stitches are joined.
- At round 1, follow the natural flow of the chain graph. This can be done by constantly holding the piece so that the small neighborhood you are currently working on is flat and untwisted, and the working yarn is in the back of that small neighborhood.

Conclusion

Understanding the design process gives us tremendous freedom to create more forms. We have covered two methods that lead us from a diagram to a chain graph that captures all topological features of the final surface. Diagrams abound in nature [5]; hence, so do the projects. These methods are by no means the only options. We look forward to future opportunities to discuss other methods in the future.

Acknowledgments

The author is hugely grateful for all the math artists who have encouraged her to continue doing what she loves: Eve Torrence, Ellie Baker, Carolyn Yackel, Chaim Goodman-Strauss, and many more. Many thanks also go to those who have attended the Topological Crochet workshops in MoMath and provided valuable feedback.

References

[1] C. C. Adams. "Section 2.4: Knots and Planar Graphs." *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*. American Mathematical Society, 2004.

[2] B. Collins. *Modular Toroids*. https://static1.bridgesmathart.org/bcollins/gallery6.html

[3] S. Dong. "Sculpting Mapping Cylinders: Seamless Crochet of Topological Surfaces." *Bridges Conference Proceedings*, Halifax, Canada, Jul. 27–31, 2023, pp. 559–566. http://archive.bridgesmathart.org/2023/bridges2023-559.html

[4] F. Herr. *Crocheting a Seifert surface | a math-y crochet project.* https://www.youtube.com/watch?v=UgoGGRhlhPU

[5] C. Livingston and A. H. Moore. *KnotInfo: Table of Knot Invariants.* https://knotinfo.math.indiana.edu/

[6] C. H. Séquin. *Part Description and Specifications for "Saddle Trefoil."* https://people.eecs.berkeley.edu/~sequin/SFF/spec.sadltref.html

[7] Wikipedia. *Saddle point*. https://en.wikipedia.org/wiki/Saddle_point

[8] J. J. van Wijk. *Visualization of Seifert Surfaces*. 2005. https://www.win.tue.nl/~vanwijk/seifertview/