How to Sew a Nine-Color Map on a Genus-Three Torus

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Abstract

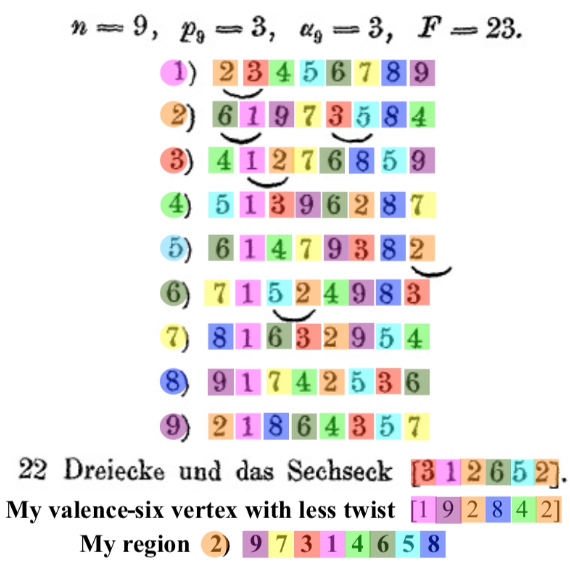
In 1891, Lothar Heffter described a complete map with nine regions on a surface of genus-three. Heffter used a table of numbers to present his map, which makes it very hard to visualize. In this workshop we will make a felt model based on Heffter's map using a slight modification to his specifications that simplifies the structure.

Introduction

In his 1891 paper [3], Lothar Heffter published a table (Figure 1(a)) that describes the layout of a nine-region map on a three-holed torus for which every region shares exactly one boundary line with every other region. In this table, each region is designated by a number, and the rows of the table give the regions that are adjacent to a region in order. I have added color to Heffter's original table to make it easier to see how the regions are distributed. Heffter also states that there are 22 vertices where three regions meet ("22 Dreieke") and one vertex where six regions meet ("das Sechseck").

In [1], Ellie Baker and I described how we made a fabric model of Heffter's map in which every region was a regular octagon. That model gave us a lot of insight into Heffter's map and confirmed that it lay on the surface of a genus-three torus. But it was tightly tangled in a way that obscured the overall structure. It was not the easy-to-understand model we had hoped for.

Seeking a better way to visualize this map, I distorted the shape of the octagonal regions until I found the design in Figure 1(b). This model was displayed at the 2023 Bridges Exhibition of Mathematical Art [7]. Sylvie Benzoni-Gavage [2], Carlos Séquin [4], and Timothy Sun [5] also shared their work on visualizing nine-color maps on a genus-three torus at Bridges 2023. And I presented a paper [8] on how to build my origami model [6] of one of Séquin's maps.



$n = 9, \quad p_9 = 3, \quad \alpha_9 = 3, \quad F = 23.$

1)	2	3	4	5	6	7	8	9
2)	6	1	9	7	3	5	8	4
3)	4	1	2	7	6	8	5	9
4)	5	1	3	9	6	2	8	7
5)	6	1	4	7	9	3	8	2
6)	7	1	5	2	4	9	8	3
7)	8	1	6	3	2	9	5	4
8)	9	1	7	4	2	5	3	6
9)	2	1	8	6	4	3	5	7

22 Dreiecke und das Sechseck [3 1 2 6 5 2].

My valence-six vertex with less twist [1 9 2 8 4 2]

My region 2) 9 7 3 1 4 6 5 8

(a)

(b)

Figure 1: *(a) Heffter's table for a nine-region complete map on a genus-three surface with color and my changes added. (b) My model of a nine-region complete map with a different valence-six vertex.*

A Twist

As I began writing this paper, I realized my model in Figure 1(b) differs in two minor ways from Heffter's description that I hadn't noticed before. Heffter states that the regions around the valence-six vertex are 3, 1, 2, 6, 5, 2 (region 2 is repeated). In my model, the regions around the valence-six vertex (marked with a blue dot in Figure 1(b)) are pink (1), purple (9), orange (2), dark blue (8), light green (4), and orange (2). I've added this vertex description below Heffter's table in Figure 1(a). This change causes the model to be less twisted. It also changes the order for the regions adjacent to region 2. This still gives a valid nine-color map with 22 valence-three vertices and one valence-six vertex. It is just not exactly the same as Heffter's map as I had previously thought.

We were able to make our model in [1] match Heffter's table because we used a stretchy fabric that accommodated the twisting. We did find the twisting frustrating and difficult to sew. I recently made a different model from the one in this paper that adheres precisely to Heffter's table. In that model (shown in the Supplement to this paper), the twisting in Heffter's map is incorporated into the shape of the red, yellow, purple, and pink regions, and results in a swirling pattern on one side of the center annulus. Perhaps if Heffter had made a physical model of his map, he would have put his valence-six vertex in a more natural location. This might have simplified earlier attempts by others to make a physical model of Heffter's map.

The Pattern

In this workshop, we will make a small version of the model in Figure 1(b). Materials needed are felt in nine colors, scissors, and a stapler. Sewing and/or stuffing the model is optional and can be done later. We will staple the seams on the outside to keep things simple and save time. The Supplement for this paper contains full-size versions of the pattern templates shown in Figures 2(a) and 2(b) that can be printed on 8.5" by 11" paper. These pattern templates are used for eight of the regions, while the ninth region is made from a strip of felt and will be explained later.

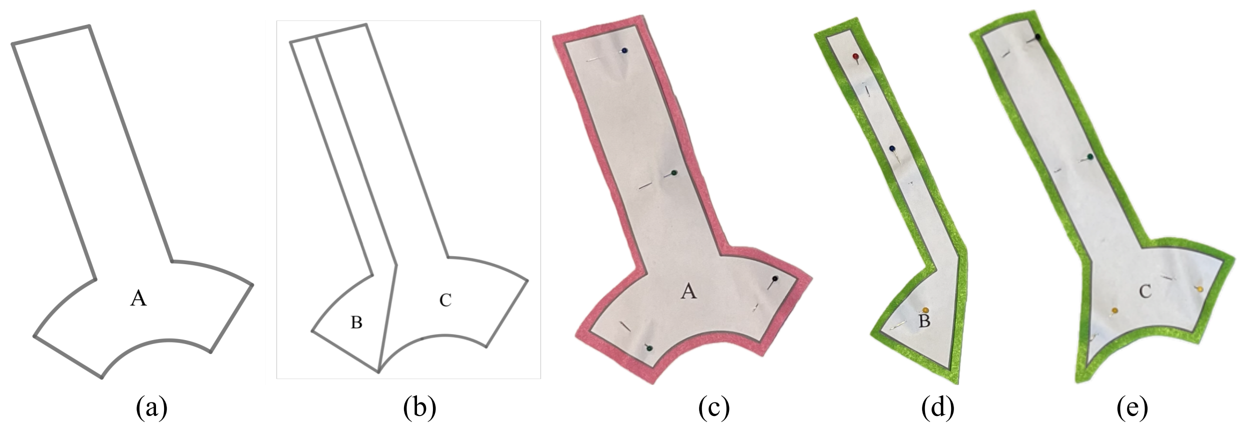


Figure 2: *(a) Pattern piece A. (b) Pattern pieces B and C. (c)–(e) Leave a one-quarter inch seam allowance around the template when cutting out felt pieces.*

Begin by cutting out the paper pattern along all the solid lines, including the line that separates the B and C pieces. The pattern template does not include a seam allowance. When cutting out the felt pieces leave a one-quarter inch seam allowance on every edge, as shown in Figures 2(c)–(e). You will need four A pieces in four different colors, and four each of the B and C pieces in four other colors, for a total of eight colors.

The images in Figure 2 show the front side of the pieces. It is important to keep track of the front and back of the felt pieces once they are cut out. I recommend marking the back of each piece with an X.

Assembling the Pieces

Join the four sets of B and C pieces of the same color, as shown in Figure 3. Then staple along the dashed lines in Figure 4(a) to join the A pieces into an annulus and in Figure 4(b) to join the B/C pieces into a separate annulus.

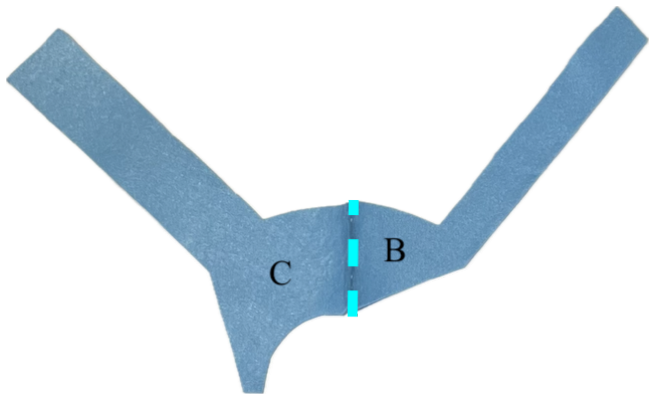


Figure 3: *Join the four sets of B and C pieces along the dashed line.*

(a) (b)

Figure 4: *(a) Join the four A pieces along the dashed lines. (b) Join the four B/C pieces along the dashed lines.*

Place the two annuli together with the backs of the pieces facing each other. Line up the arms and then staple all the dashed lines shown in Figure 5. Once the annuli are joined, you will notice the arms start to naturally twist, as shown in Figures 6(a) and 6(b). The arms are cut at an angle that causes these twists when they are straightened. We will soon see these twists are necessary to allow every region to share a boundary with every other region.

Note that each arm has an A, B, and C piece. The A part of each arm is three units wide, the B part of each arm is one unit wide, and the C part of each arm is two units wide. Figure 6(a) shows the A side of the joined annuli with the arms straightened. At the end of each arm on this side, the B piece should be flat, and the A piece should be folded at two-thirds of its width. Figure 6(b) shows the B/C side of the joined annuli with the arms straightened. At the end of each arm on this side, the C piece should be flat, and the A piece should be folded at one-third of its width. This alignment will be crucial when we join the arms into tubes. But first, we will add the ninth region to close the center of the annulus.

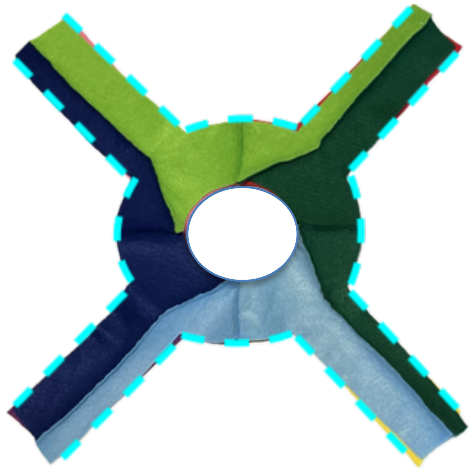


Figure 5: *Staple the two annuli together along the dashed lines.*

(a) (b)

Figure 6: *(a) The A side of the joined annuli. (b) The B/C side of the joined annuli.*

The Ninth Region is Special

We could close the center hole by adding a short cylinder of felt with the top of the cylinder attached to the A side of the hole, and the bottom of the cylinder attached to the B/C side of the hole. But the rules of map coloring require every region of a map to be a topological disk, meaning no holes are allowed in a region. A cylinder is a topological annulus. To fix this problem, Heffter pinches the cylinder together at one point so this region meets itself at just one point, and is thus a topological disk.

Cut a strip of felt in the ninth color that is 1.5 inches by 16 inches. You may need to staple two strips together to make it long enough. Taper one end of the strip so it is half an inch wide, like the orange strip in Figure 7(a). Once the strip is completely attached, with a one-quarter inch seam allowance on each edge, the end of the strip will taper down to a point.

Attach the tapered end at a place where two regions meet on the A side of the annulus. Staple one side of the strip to the A side of the annulus hole. Once you have stapled around the entire A side circle, taper the other end of the strip to match the starting end. Staple the other side of the strip to the B/C side of the annulus hole, making sure that where the tapered ends meet you have a vertex with valence-six, as in Figure 7(b).

Note that there are four places at the center edge of the annulus where two A regions and two B/C regions can meet at a point. We can use any of these four points as the valence-six vertex when we join the ninth region. In Figure 7(b), the valence-six vertex is adjacent to the purple, yellow, orange, light blue, and dark blue regions. This is a different vertex than in my model in Figure 1(b), but this still forms a valid nine-color map with 22 valence-three vertices and one valence-six vertex.

(a) (b)

Figure 7: *(a) Join the tapered edge of the ninth region to the annulus hole at a place where two regions meet on the A side. (b) Once the ninth region is attached there should be a valence-six vertex on the inside of the annulus hole.*

The last step is to join opposite arms into two tubes, one on each side of the center annulus. This must be done very carefully so that in the final model every color region shares a boundary line with every other region. Many of these shared boundary lines occur where the arms are joined to each other. If the arms are in the positions shown in Figures 6(a) and 6(b), then when they are brought together, the B part of each arm will line up with the middle third of the A part of the opposite arm. Figure 8 shows the attached tubes on the two sides of the surface. The diagrams in Figure 9 depict the correct arrangement of the boundaries for the corresponding tube in Figure 8.

Figure 8: *Check that the ends of the arms are properly aligned before attaching.*

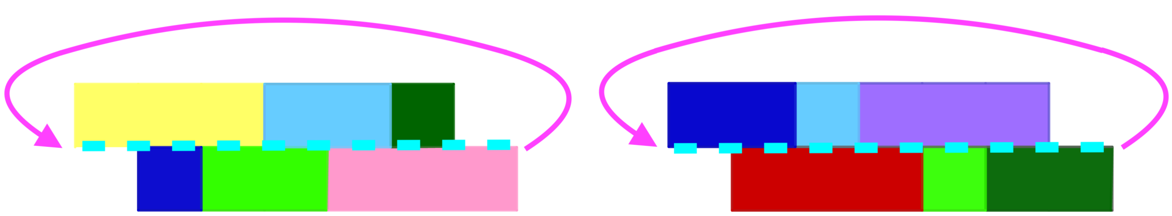


Figure 9: *The arrangement of the regions at the end of the arms where they connect along the dashed lines to form tubes.*

Summary

Once your model is finished, you can check that it is indeed a nine-color map on a genus-three surface. The three holes of the surface are clearly visible. You can trace along the boundary of each region to see that it has eight different neighbors. Since there are only three types of regions, and the model is highly symmetric, you only need to check one of the A regions, one of the B/C regions, and the ninth region to see you have a complete map.

Figure 10: *A stapled workshop model (left) and a machine-sewn and stuffed model (right).*

References

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