Regular Arrangements of Platonic Solids

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Abstract

We explore various ways to assemble Platonic solids into regular structures, hoping to inspire artists and designers.

Introduction

We investigate each type of polyhedron individually; in other words, each structure only contains one type of Platonic solid. We will look for both chain-like structures, which can be interpreted as one-dimensional, and three-dimensional structures. Our aim is not to fill a space but rather to obtain any type of uniform and regular structure consisting of one type of Platonic solid. By regularity we also mean that all the solids forming the regular structure must have the same number of joins in the same places. We are looking for infinite structures, but below we also mention some examples of finite regular structures. We present examples of different solutions, not a complete list. The individual sectors of our topic are so common that several of the structures we present have been presented previously in various contexts.

Types of joins

Polyhedra can be regularly joined together face to face, edge to edge or vertex to vertex. We do not investigate the less regular joins, such as face to edge, face to vertex or edge to vertex. The most regular joins are, as follows:

Faces can join together in two regular ways: either with the joined faces coinciding or with one of the faces rotated by half of the polygon's central angle with respect to the other face.

Edges can either coincide or only join at their centers. In the latter case, the edges are rotated by 90° with respect to each other. In both cases, the centers of the joined solids and the centers of the joined edges are, all four, on the same line.

Vertices join together with the centers of the joined solids and the joining vertices all on the same line. The vertex figures of the joined vertices must be oriented to each other like the faces in a face-to-face join.

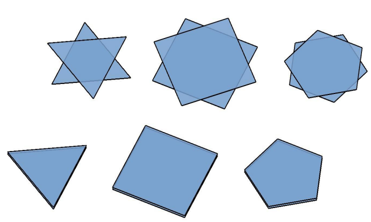

Figure 1: *Face-to-face*

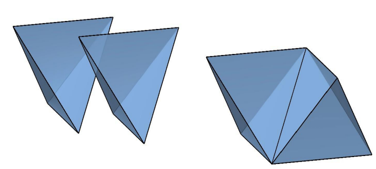

Figure 2: *Tetrahedra, edge to edge*

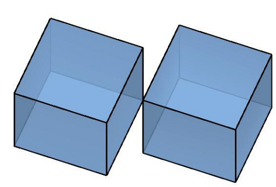

Figure 3: *Cubes, vertex to vertex*

If a solid only has one active join (face, edge or vertex), the solids can only be joined in pairs. If a solid has two active joins, the solids can only be stacked into chain-like arrangements, either infinite lines or spirals or finite rings. If there are three active joins, the structure will become tree-like. If there are four or more, the structure can be completely symmetrical.

Chains

By a chain, we mean a structure in which each solid is only joined to two others. The edges of all Platonic solids are arranged in opposing pairs, and thus they can all be assembled edge-to-edge into straight chains. Similarly, the faces and vertices of hexa- octa-, icosa- and dodecahedra are also arranged in opposing pairs, so they can all be assembled into straight chains face to face or vertex to vertex.

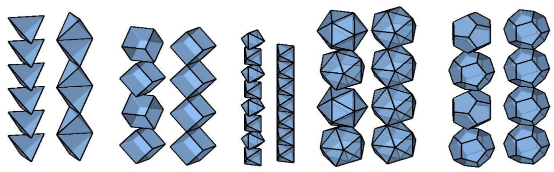

Figure 4: *Edge-to-edge chains*

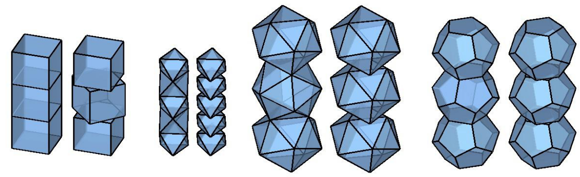

Figure 5: *Hexa- octa-, icosa- and dodecahedra assembled into face-to-face chains*

Tetrahedra have a vertex and a face opposite each other, so tetrahedra can only be assembled into helical chains. [1] [5]

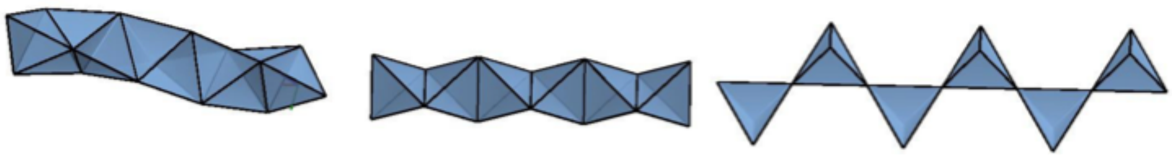

Figure 6: *Helical chains of tetrahedra, face to face, edge to edge, and vertex to vertex*

Icosahedra can also be assembled into helical chains. Likewise, dodecahedra can be assembled into helical chains. [11] [12]

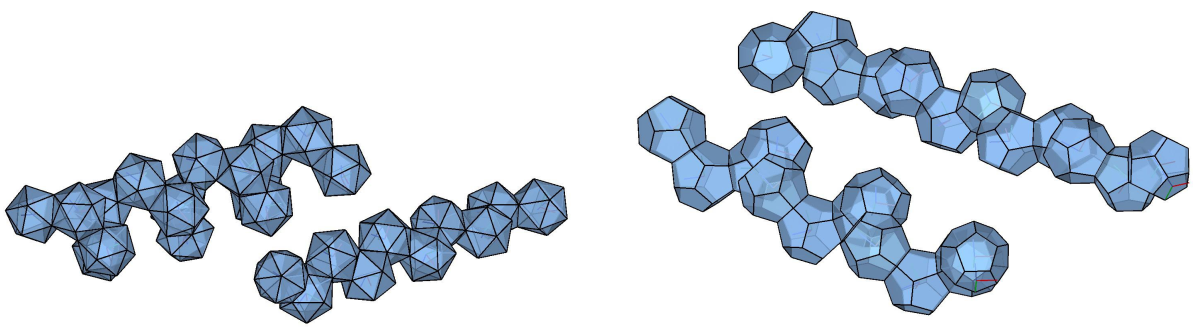

Figure 7: *Helical chains of icosahedra and dodecahedra, both face to face*

Six tetrahedra can be assembled into a vertex-to-vertex ring. Four and six cubes can be assembled into vertex-to-vertex rings. Six icosahedra can be assembled into an edge-to-edge ring. Five and ten dodecahedra can be assembled into edge-to-edge rings. [5] [6]

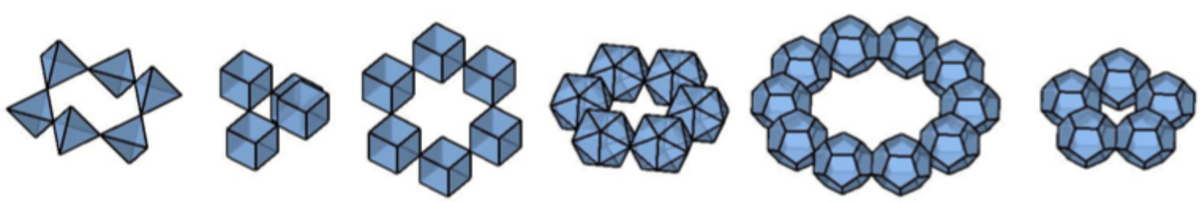

Figure 8: Examples of rings

Infinite Polyhedral 3D-Arrangements

Tetrahedra can be joined edge to edge in the same way as tetrahedra and octahedra can be combined to fill up a space: removing the octahedra results in tetrahedra joined edge to edge.

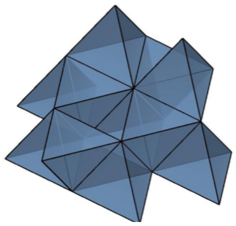

Figure 9: *Tetrahedra joined edge to edge, with edges coinciding.*

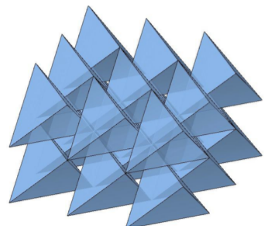

Figure 10: *Tetrahedra edge to edge, meeting edges perpendicular*

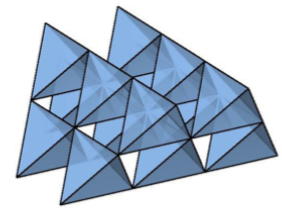

Figure11: *Tetrahedra vertex to vertex*

Cubes can be stacked face to face to fill the space. Cubes can also be stacked edge to edge or vertex to vertex (Figures 12 and 13).

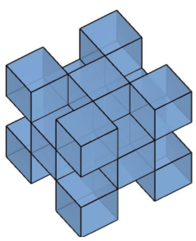

Figure 12 *Cubic structure edge to edge*

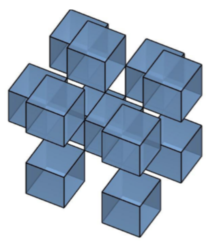

Figure13: *Cubic structure vertex to vertex*

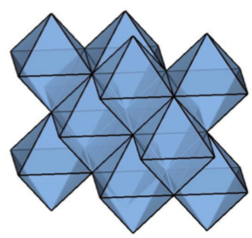

Figure 14: *Octahedral structure edge to edge*

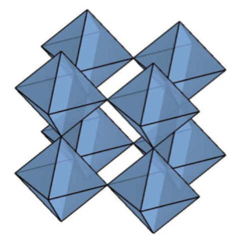

Figure 15: *Octahedral structure vertex to vertex*

Octahedra (Figures 14 and 15) can be joined edge to edge the same way as the tetrahedra described above, but removing the tetrahedra from the space-filling tetra-octa structure.

Joining octahedra vertex to vertex is simple. The octahedra are placed inside their circumcubes (the smallest cube which contains the octahedron) and the cubes are stacked to fill up the space, after which the cubes are removed. [4]

Icosahedra (Figures 16 and 17) can be joined face to face. However, only the six faces (out of all twenty) whose center points meet the vertices of the icosahedron's incube (the greatest cube inside the icosahedron), can be joined to other faces. [4] Similarly, icosahedra can be joined edge to edge, but only at the six edges which are on the faces of the icosahedron's circumcube.

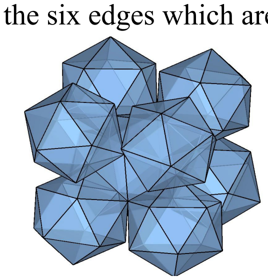

Figure 16: *Icosahedral. structure face to face.*

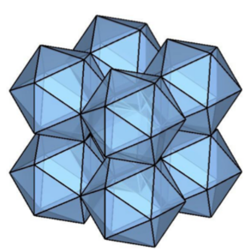

Figure 17: *Icosahedral structure edge to edge*

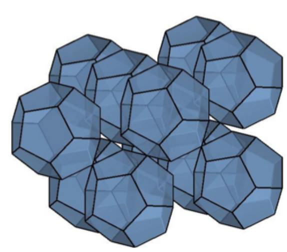

Figure 18: *Dodecahedral structure vertex to vertex*

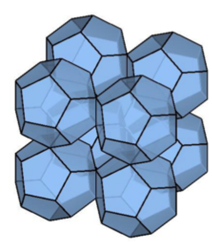

Figure 19: *Dodecahedral structure edge to edge*

Dodecahedra (Figures 18 and 19) can be joined vertex to vertex at the eight vertices which are vertices of the dodecahedron's incube. [4] Similarly, dodecahedra can be joined edge to edge, but only at the six edges which are on the faces of the dodecahedron's circumcube. [4]

Finite Polyhedral Arrangements

We present one example. Twelve dodecahedra can be assembled into a spherical shape, in similar order as the pentagonal faces of a dodecahedron. The midpoints of the 12 dodecahedra are located as vertices of an icosahedron. The dodecahedra touch edge to edge, each of those joined to others at five edges.

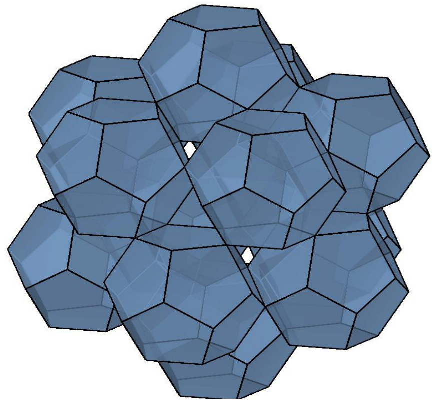

Figure 20: *12 dodecahedra edge to edge*

Summary and Conclusions

We feel that a good approach to our topic is the way children play with wooden blocks: with the mind open to anything and everything. A similar kind of openness is often fruitful in art, architecture and design. Examples of this are, of course, numerous. Here are a few: [2] [3] [6] [7] [8] [9] [11].

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