Polyhedral Cages: An Aesthetic Generalisation Of Regular Solids

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Abstract

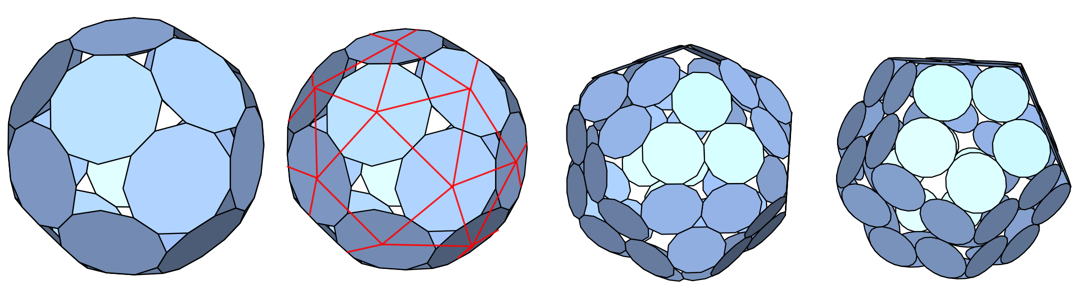
Polyhedral cages are assemblies of regular or nearly regular polygons, sharing some of their edges, but otherwise separated by holes. After describing them, we present a number of symmetric examples to illustrate how aesthetic many of them are, suggesting potential applications in arts, craft works or architecture.

Introduction

In geometry several convex *regular solids* have been known since the antiquity. Five convex regular (Platonic) solids have been known since the antiquity. These are solids made out of regular and identical polygons so that all the vertices are identical. There are only five of them: tetrahedron, cube, octahedron, dodecahedron and icosahedron. Solids made of regular (but not just one type of) polygons are the 13 Archimedean solids, prisms and anti-prisms [1].

More recently, Johnson [3][12] defined and identified 92 convex solids made out of any number of regular polygons with no conditions on the vertices. *Near-miss Johnson solids* are defined as solids made out of nearly regular polygons. While there are many examples, so far they have not been classified [5].

Polyhedral cages (p-cages for short) are a generalisation of regular solids [7], as assemblies of regular planar polygons such that the polygons share at least 3 edges with adjacent polygons but with the condition that if two edges of a face are adjacent to each others only one of them can be shared with another face, hence making these *non shared* edges part of a hole. In what follows, we are only interested in convex p-cages. *Near-miss p-cages* are p-cages made out of nearly regular polygons. (See Figure 1 for examples of p-cages).



(a) (b) (c) (d)

Figure 1: *(a) Near-miss p-cage made out of 24 hendecagons. Underlying symmetry: snub cube. (b) The red lines define the hole-polyhedron. (c) 30 12-gons, (d) 30 20-gons.*

The main difference between solids and p-cages is that p-cages have holes which are not required to be regular or planar in any way. As the number of potential p-cages is very large we restrict ourselves to p-cages that have some symmetries.

A p-cage is said to be *homogeneous symmetric* [7][9] if all the faces are identical and *equivalent* so that for any pair of faces there is a rotation, excluding reflections, of the p-cage that maps one face onto the other face as well as the entire p-cage onto itself. A p-cage is said to be *bi-homogeneous symmetric* [10] if it is made out of 2 families of faces such that all the faces are equivalent to all the other faces belonging to the same family. This means that for any pair of faces belonging to the same family there is a rotation of the p-cage that maps one face onto the other face as well as the entire p-cage onto itself.

The first p-cage structure was discovered as an artificial protein cage [8]. As it turns out, many of the p-cages happen to have quite aesthetic properties and could be used in the arts, in architecture or to design objects such as jewellery, lampshades or science fiction spacecrafts. Regular or nearly regular objects have already been created in mediums such as beadwork[11], jewellery[6], wood [2] or 3d-printed sculpture [4].

Constructions

Constructing p-cages can be done by trial and error, but by imposing some symmetry one can determine mathematically a constrained range of possible coordinates for the p-cage faces. One must then find the coordinates for which the p-cage is regular, or nearly regular. First we observe that if we draw the lines joining the centres of the faces of a p-cage that share an edge, we obtain a solid that is, by definition [7], the *hole-polyhedron* of the cage, Figure 1(b). Being a polyhedron, the hole-polyhedron corresponds to a planar graph and it describes how the faces of the p-cage are linked together. The vertices and the faces of the hole-polyhedron correspond respectively to the faces and the holes of the p-cage, as shown in Figure 1(b).

Imposing that the faces of the p-cage are equivalent then corresponds to imposing that the vertices of the hole-polyhedron are equivalent. For the homogeneous symmetric p-cages such equivalent graphs are well known [1][7] and correspond to the Platonic solids, Archimedean solids, prisms and anti-prisms. Their symmetry groups are finite subgroups of the orthogonal group of 3 space. For the p-cage faces to be equivalent modulo a proper rotation of the p-cage we must disregard the truncated cuboctahedron and the truncated icosidodecahedron. The symmetry of the p-cages constructed from these graphs will be a sub-group of the symmetry group of the associated graph, *i.e.*, a subgroup of the symmetries of the regular solids.

The number of possible hole-polyhedron graphs for bi-homogeneous p-cages is very large so in [10] we have restricted ourselves to a small subset. Given any planar graph we can construct a p-cage by choosing a polygon (two for the bi-homogeneous p-cages) and decide which of the polygon edges will be shared by the neighbours. There are many polygons to chose from and many ways to attach these polygons together. Figures 1(c) and 1(d) provide two examples of p-cages derived from the same graph. The symmetry of the hole-polyhedron graph imposes some restrictions on how this can be done but the number of p-cages so obtained is very large, hence we must restrict ourselves to p-cages which are only slightly irregular.

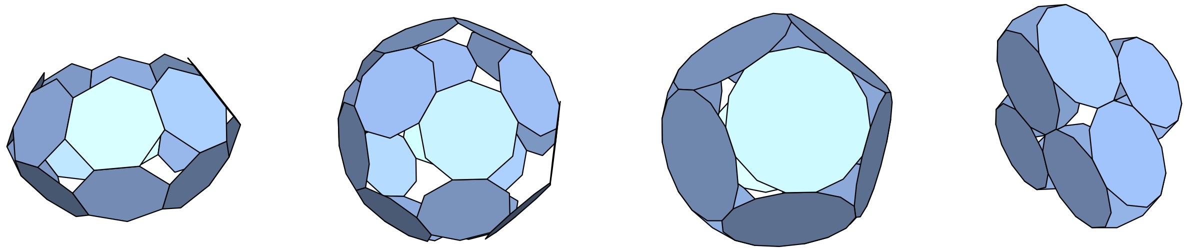
As the size of a p-cage is arbitrary we can set its edge length of a regular polygon to 1 without any loss of generality. As the angle between adjacent face edges of a regular P-gon is $\alpha = \pi(1 - \frac{2}{P})$, then if d_i is the length of the edges of all the p-cage faces and α_i the angle between adjacent edges, the p-cage deformation of the length, Δ_l, the angle, Δ_a, and the overall relative deformation, Δ, are given by

$$\Delta_l = \max_i(|\frac{d_i - 1}{1}|), \qquad \Delta_a = \max_i(|\frac{\alpha_i - \pi(1 - \frac{2}{P})}{\pi(1 - \frac{2}{P})}|), \qquad \Delta = \max(\Delta_l, \Delta_a). \tag{1}$$

Results

For nearly all hole-polyhedron graphs there are some regular p-cages and the coordinates of their faces can be derived geometrically. Some examples are shown in Figure 2.

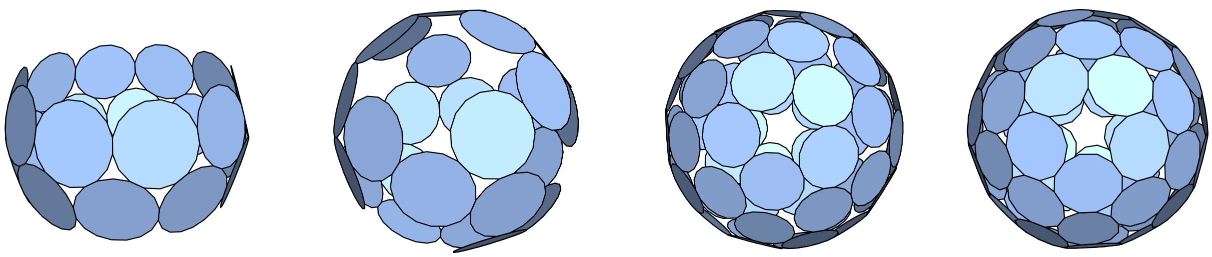
P-cages can have a range of overall shapes; some are sphere-like, such as in Figures 1(a) and 2(b).



(a) (b) (c) (d)

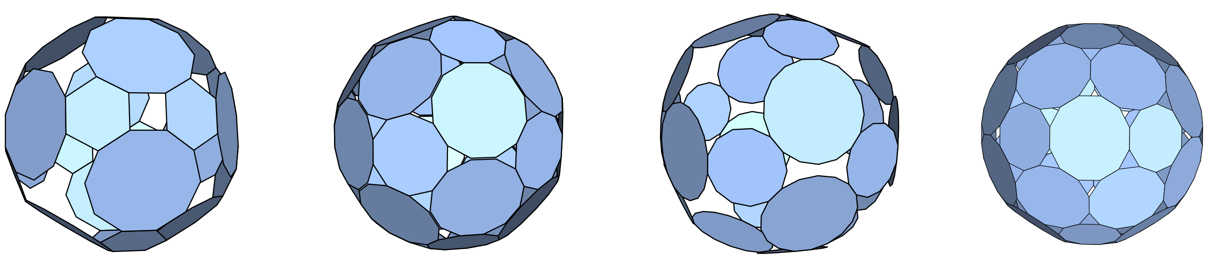
Figure 2: *Regular p-cages (a) 16 heptagons; (b) 20 nanogons; (c) 12 15-gons; (d) 12 12-gons.*

P-cages generated from the prism and anti-prism usually have a ring-like shape, as shown in Figure 2(a) and 3(a). Others have box like shapes such as in Figures 1(c), 1(d) and 2(d). Another characteristic of the p-cages are the shapes of the holes. Some can be quite large (Figure 3(b)) or small (Figure 2(c)). Some of the holes are star or polygonal-like in shape (Figures 1(a), 2(c) and 3(d)) while others look like thin filaments (Figures 3(c)).



(a) (b) (c) (d)

Figure 3: *Near-miss p-cages (a) 20 17-gons ($\Delta = 0.38\%$); (b) 24 17-gons ($\Delta = 0.06\%$); (c) 60 16-gons ($\Delta = 0.042\%$); (d) 60 20-gons ($\Delta = 3\%$).*



(a) (b) (c) (d)

Figure 4: *Bi-homogeneous p-cages, regular: (a) 12 hexagons, 12 12-gons; (b) 12 nanogons, 20 12-gons; Near-miss: (c) 12 13-gons, 12 17-gons ($\Delta = 0.007\%$) ; (d) 12 10-gons, 30 12-gons ($\Delta = 1.7\%$).*

Most homogeneous p-cages have relatively large holes. In contrast, bi-homogeneous p-cages have more varied geometries and, in many cases, small holes. Examples of such cages are shown in Figure 4. Figure 4(a) is an example of a p-cage made out of two very different polygons. Notice that the regular p-cage in

Figure 4(b) has some nearly touching faces which can be merged by deforming the faces slightly further. The faces of p-cages derived for equivalent graphs have at most 5 neighbours each, but most have only 3 or 4 neighbours. Faces of bi-homogeneous p-cages, on the other hand, can have up to 6 neighbours. The p-cage shown in Figure 4(d) has the largest possible connectivity between faces: 6 and 5 neighbours for respectively the dodecagons and the decagons. If one fills the triangular holes, one obtains a near-miss Johnson solid. Some near-miss p-cages can be very close to being regular as shown in Figure 4(c) for which the overall relative deformation is $\Delta = 0.007\%$.

Summary And Conclusions

In this paper, we have considered polyhedral cages, *i.e.*, assemblies of polygons, regular or nearly regular, containing holes, based on [7][9][10]. We restricted ourselves to p-cages that are symmetric, leading to a number of structures which are either ring shaped, or spherical. Interestingly, some near-miss p-cages are made out of polygons not found in any of the Platonic or Archimedean solids, including polygons with odd numbers of edges (other than the triangles and pentagons), including 13-gons and 17-gons (Figure 4(c)).

Acknowledgements

This research was funded by the Leverhulme Trust Research Project Grants RPG-2020-306.

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