A Modular Sculpture Corresponding to Three Rotations

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Abstract

A modular sculpture with simple and highly symmetric units is observed to have no apparent large scale symmetries. We show that the relative rotations of units in the sculpture are dense in the three dimensional rotation group, and ask whether they admit the presentation $\langle a, b, c \mid a^2 = b^2 = c^2 = 1\rangle$.

Introduction

In 2022 the first author made *Flowermountain*, a modular sculpture comprised of Douglas fir units, as shown in Figure 1. Each unit is a rectangle with slits, so they could slide together and hold in place, with any two interlocked units lying in orthogonal planes. The slits were made at regular angles; they were any of $0°, 60°, 120°, 180°, 240°$, or $300°$ as shown in Figure 2. The units were hand-made, resulting in imprecisions in the angles, and there were further liberties in choosing where slits were positioned. Moreover, units were one of three sizes. Due to the regularity in the angles of the slits, the first author expected *Flowermountain* to exhibit some large scale symmetries–for example, maybe some distantly positioned units would lie in parallel planes. However, he was surprised to see that the result was highly irregular with no visible symmetry.

Figure 1: *Flowermountain, 2022.*

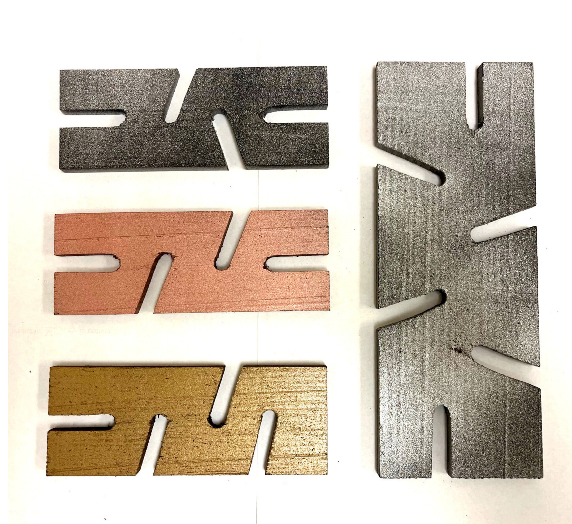

Figure 2: *Examples of units in Flowermountain.*

In this paper we explore a new sculpture, shown in Figure 3 and Figure 4, inspired by *Flowermountain*. The idea was to create a modular structure which removes the imprecisions and human choices in *Flowermountain* described above, and observe whether or not the apparent lack of symmetries persists. The new sculpture consists of identical 3d-printed units, each with three orthogonal slits spaced 120° apart. In the next section we quantitatively describe how the interlocked units are related to the base unit, which allows us to use group theory for further exploration of the geometry of the sculpture. Using this, we roughly show that for any choice of rotation, there is a chain of interlocked units so that the first and last units are related by the rotation to arbitrary precision. Finally we explore the question of "which symmetries manifest in the sculpture?" This corresponds to investigating whether there are any relationships between the rotations of units in the sculpture.

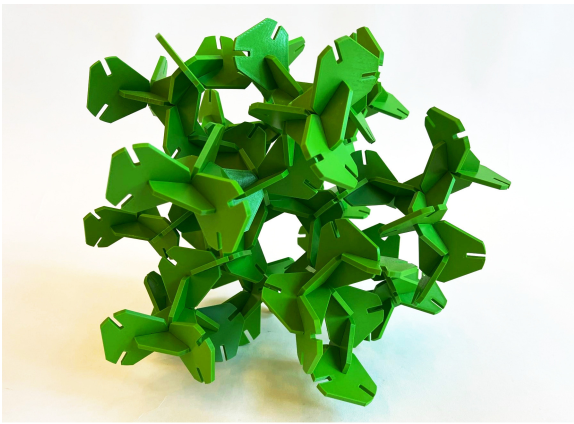

Figure 3: *An abstraction of Flowermountain comprised of 3d-printed units.*

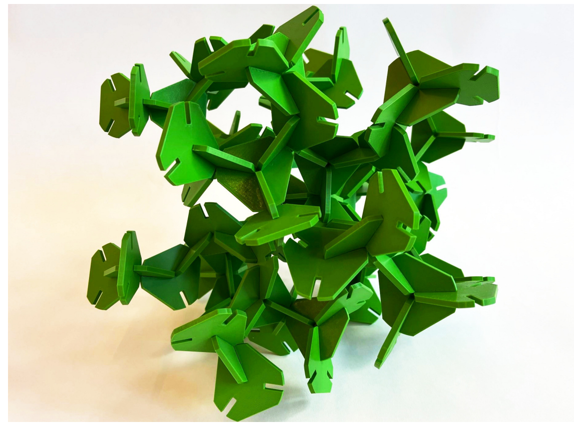

Figure 4: *A different view of the modular sculpture.*

In What Sense is the Sculpture Asymmetric?

Figure 3 and Figure 4 show the result of building the modular sculpture as "symmetrically as possible". We started with two units in the center, then interatively added four units, eight units, and so on. At several stages of this process, units couldn't be added as illustrated in Figure 6, but these conflicts also arose symmetrically around the sculpture. Figure 5 shows the first few steps of this procedure. We considered an alternative procedure starting with a central unit which technically results in slightly more rotational symmetries, but the result was visually less appealing.

Despite building the sculpture as symmetrically as possible, there are no apparent translational symmetries. However, the particular construction in Figure 3 and Figure 4 exhibits global rotational symmetry (with symmetry group the Klein four-group $(\mathbb{Z}/2\mathbb{Z})^2$). This is analogous to Penrose tilings which have no translational symmetry but exhibit five-fold rotational symmetry [2] as shown in Figure 9.

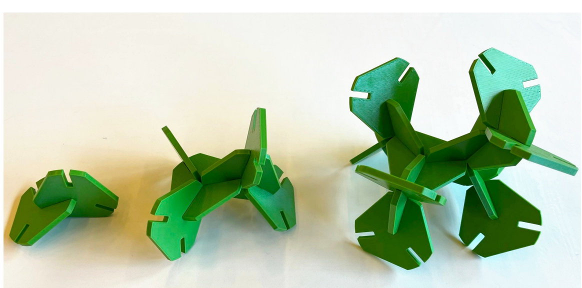

Figure 5: *An iterative process for maximizing symmetry.*

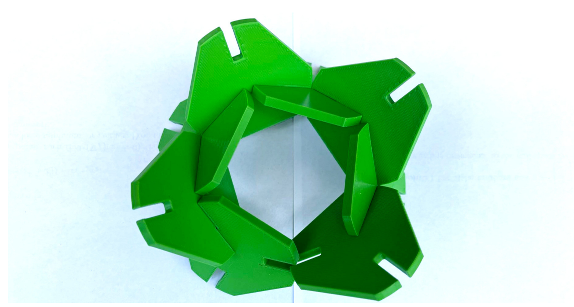

Figure 6: *A unit cannot be added to either of the adjacent slits at the bottom.*

The Sculpture as a Group of Rotations

Each unit can interlock with another at its three vertices. Figure 8 shows a unit in the (x, y)-plane with vertices a, b, and c, and another unit with vertices a', b', and c'. There is a unique affine transformation—that is, a rotation followed by a translation—sending a to a'; ignoring the translation, the rotation is given precisely by the matrix A below. Similarly, there are unique affine transformations with corresponding rotation matrices B and C (also given below) that interlock b with b' and c with c', respectively.

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \frac{1}{4}\begin{pmatrix} -1 & -\sqrt{3} & 2\sqrt{3} \\ -\sqrt{3} & -3 & -2 \\ 2\sqrt{3} & -2 & 0 \end{pmatrix}, \quad C = \frac{1}{4}\begin{pmatrix} -1 & \sqrt{3} & -2\sqrt{3} \\ \sqrt{3} & -3 & -2 \\ -2\sqrt{3} & -2 & 0 \end{pmatrix}.$$

Each of these matrices is an involution, meaning they square to the identity matrix. This can be seen geometrically because they are each 180° rotations about axes depicted in Figure 7. The axes are the three diagonals of an *antiprism* of an equilateral triangle, whose side-faces have base to leg ratios of $\sqrt{3}$ to $\sqrt{5}$.

There is a correspondence between words over $\{A, B, C\}$ and sequences of units in the sculpture as follows. First, any two consecutive units are related by one of the three rotations A, B, or C. A sequence of these rotations forms a word such as $CA \ldots ABACB$. The resulting word, interpreted as a product of matrices, describes the rotation sending the first unit to the last unit in a chain of interlocked units. Conversely, any given word over $\{A, B, C\}$ is a sequence of rotations to apply when building a chain of interlocked units, with the caveat that a chain might loop back and try to intersect itself as shown in Figure 6.

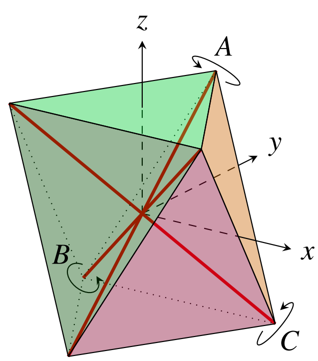

Figure 7: *Axes of rotation of* A, B, *and* C.

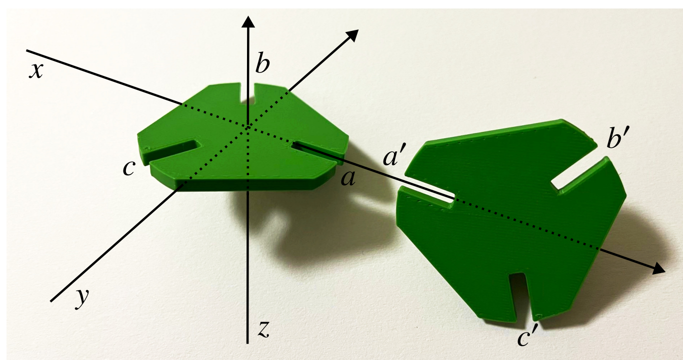

Figure 8: *Setup to determine rotation matrix* A.

Rotations of the Sculpture are Dense in $SO_3(\mathbb{R})$

In this section we prove that $\langle A, B, C\rangle$, the group of rotations of sculptural units, is dense in the group $SO_3(\mathbb{R})$ of all 3-dimensional rotations. This mathematical statement has the following sculptural consequence: given any rotation, there is a chain of interlocked units such that the first and last units are related by a rotation which approximates the given rotation (to arbitrary non-zero precision). Again, the caveat is that not all sequences can be realized, as in Figure 6.

Proposition. *The group* $\langle A, B, C\rangle$ *is dense in* $SO_3(\mathbb{R})$.

Proof idea. Consider an arbitrary rotation matrix R with axis K and angle of rotation θ. We will show that words over $\{BA, AC\}$ can approximate R. First we note that BA and AC are rotations by irrational angles about distinct axes L_1 and L_2. Consequently, iterations of BA are dense in the rotations about L_1, as are iterations of AC about L_2. It follows that there is a word W over $\{BA, AC\}$ sending the axis L_1 arbitrarily close to K. Next, we observe that words of the form $W(BA)^n W^{-1}$ are rotations about an axis approximating K, and choosing appropriate n, the angle of rotation θ can also be approximated. This shows that the group generated by BA and AC is dense in $\mathrm{SO}_3(\mathbb{R})$, so in particular the group $\langle A, B, C\rangle$ is dense.

Relationships Between the Rotations

The rotation matrices A, B, and C all have order 2 in the group $\mathrm{SO}_3(\mathbb{R})$. After experimenting with the units of the sculpture and the rotation matrices A, B, and C, it is natural to wonder whether any other relationships hold between these matrices, other than what can be deduced from the three having order 2.

Question. *Is the subgroup of* $\mathrm{SO}_3(\mathbb{R})$ *generated by A, B, and C a free product*

$$\langle A, B, C\rangle = \langle A\rangle * \langle B\rangle * \langle C\rangle \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}?$$

As far as the authors know this question is open. If the answer to this question is yes, then it has the following interesting consequence for the sculpture: it would mean that there is no sequence of interlocked units that returns to its starting angle. Consequently, it would be impossible to build any closed loops. It would also imply that the sculpture has no translational symmetry, even if we allow for self intersections. This contrasts with the sculpture *Bamboozle* [3][4] shown in Figure 10, in which the slits are not orthogonal to the faces of the units allowing for closed loops of size 10. Evidently there is also translational symmetry.

Using a computer we were able to check that there are no words in A, B, and C with lengths from 1 to 40 that are unexpectedly equal to the identity matrix. Allcock and Dolgachev [1] showed that the subgroup G of SO(3) generated by 180° rotations about the four diagonals of a cube as shown in Figure 11 generate a free product $*^4(\mathbb{Z}/2\mathbb{Z})$. Although G is not discrete in SO(3), it is discrete when reframed as a subgroup of $\mathrm{PGL}_2(\mathbb{Q}_3)$, and this allowed them to determine its group structure. Unfortunately, we were unable to adapt their approach to our setting, because we could only find non-discrete embeddings of $\langle A, B, C\rangle$ in $\mathrm{PGL}_2(\mathbb{Q}_p)$.

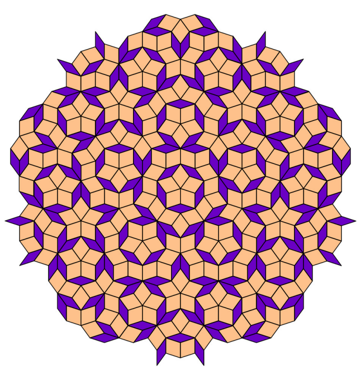

Figure 9: *A Penrose tiling.*

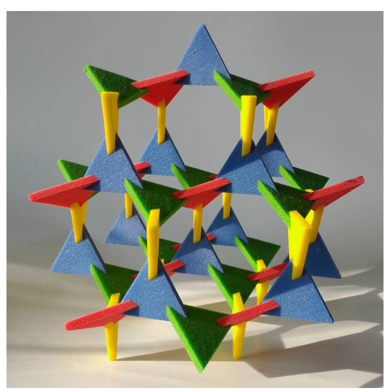

Figure 10: *Bamboozle, 2012.*

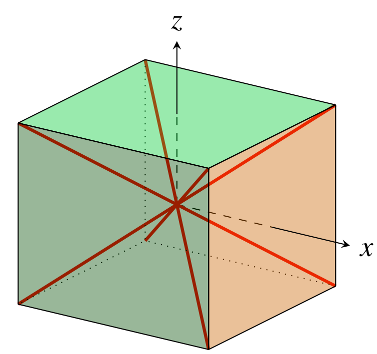

Figure 11: *Cube diagonals.*

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