Eden Model for Pentagons

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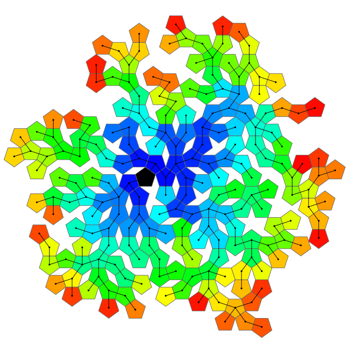
Abstract

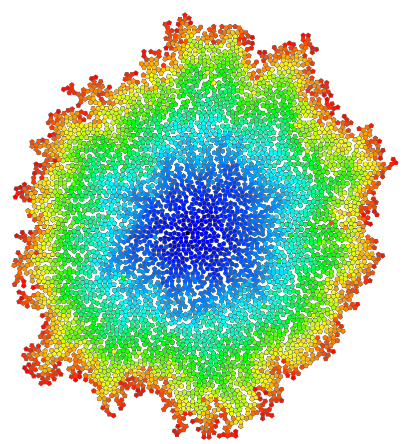
We introduce and study the topological and geometric properties of a cell growth process in the Euclidean plane, where the cells are regular pentagons. An artistic activation of this model enables a participatory and generative installation of random asymmetries and an infinite number of holes that remain uncovered.

The Model

The model involves randomly attaching regular pentagons along their edges, ensuring no overlap between any two pentagons. More precisely, let P_0 be a regular pentagon centered at the origin. Assume that n-th pentagon P_n has been positioned. Define $F(n)$ as the set of *'Free sides'*, comprising the pentagon sides s where attaching a new pentagon to s avoids overlap with any of $P_0, \ldots, P_n$. Then, a side is randomly chosen from $F(n)$ and a new pentagon P_{n+1} is attached along it, forming the $(n+1)$-th instance of the model. Subsequently, the set of Free sides is updated: $F(n+1)$ includes $F(n)$ along with the new sides of the new pentagon, minus the sides that might now cause overlap with $P_0, \ldots, P_n, P_{n+1}$. This random process is inspired by the Eden Model which is a First Passage Percolation Process (FPP) based on the regular square tessellation of the plane [2]. Recently [1, 3, 4], people have studied the topology of FPP processes based on lattices. Simulations of the Eden Model are available, see [3]. In the pentagon model, we do not have an underlying lattice and there will be holes that become impossible to cover at any future time. One of the consequences of this fact is that in the pentagon model, there is also no notion of a convex limiting shape. Moreover, we can observe from our simulations that the perimeter and the number of holes grow linearly with respect to the number of tiles, which differs from the behavior of the classical FPP models based on lattices.



(a) *Cell growth model with 200 pentagons*



(b) *Model with 5000 pentagons*

Figure 1: *The black pentagon is centered at the origin. The colors of the tiles reflect the stage at which that pentagon was placed. The underlying tree represents how the pentagons were attached.*

Geometrical and Topological Properties

A simple non-intuitive fact is that pentagons come only in two orientations, the first one and its "reflection."

Lemma 0.1. *Let P_0 be the original pentagon and P_1 a reflection of P_0 over one side. Then, all pentagons in the model are translations of P_0 or P_1. Moreover, the sides of all pentagons intersect at angles that are multiples of* $36°$.

Proof. Let L be the set of five lines through the origin which are parallel to the sides of P_0. The lines in L meet at angles that are multiples of $36°$. When a new pentagon is glued, it is obtained by reflecting over the gluing side. The angles do not change under reflection, and one side remains fixed. Therefore, the sides of any new pentagon are still parallel to the lines in L. We conclude that all the pentagons in the model have sides parallel to L and therefore can only be in two possible orientations. Moreover, any two sides of any two pentagons must intersect at the same angles as two lines in L, that is, at angles that are multiples of $36°$. □

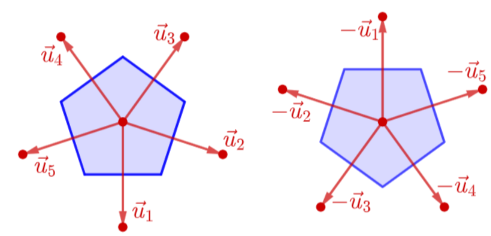


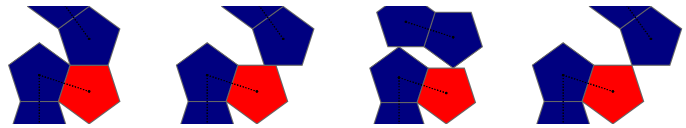
Figure 2: *Orientations and directional vectors of possible neighboring centers.*

We proceed to analyze the locus of centers of possible pentagons within the model. The center of a new possible pentagon can be obtained by translating the center of the old pentagon along 5 vectors, see Figure 2. We denote these vectors by $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5$. These can be assumed to be unit vectors, but all of our analysis is independent of scale. Let us regard $\{\vec{u}_1, \vec{u}_2\}$ as a basis for $\mathbb{R}^2$. Then $\vec{u}_3, = -\vec{u}_1 + \phi\vec{u}_2, \vec{u}_4 = -\phi\vec{u}_1 - \phi\vec{u}_2$, and $\vec{u}_5 = \phi\vec{u}_1 - \vec{u}_2$, where $\phi = 2\cos(2\pi/5) = \frac{\sqrt{5}-1}{2}$. Since ϕ is irrational, the usual argument shows that the set of integer linear combinations of the vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5$ form a dense subset of $\mathbb{R}^2$. Let r be the distance from the center of a pentagon to one of its vertices. We conjecture that the set of possible centers of the pentagons in the model is dense outside a disk of radius $2r$. To prove this conjecture we need to consider the non-overlapping geometric constraint of the pentagons. This difficulty might be overcome by choosing an appropriate geometric realization of the linear combination by deciding a plausible order in which the pentagons are glued. Additionally, a linear combination remains unchanged when we add a term $+\vec{u}_i - \vec{u}_i$. Geometrically, this corresponds to adding a pair of pentagons represented by opposite vectors $\vec{u}_i$ and $-\vec{u}_i$, in this way the linear combination remains unchanged but the geometric realization has more freedom. These two choices should give us enough maneuverability to avoid overlapping when a linear combination is transformed back to pentagons.

The Graph

In what follows, we study properties and the growth of the number of vertices, edges, and holes of the graph formed by the pentagons. We will distinguish between the *sides* of the pentagons and the *edges* of the graph. A side could be split into 2 edges. For example, in Figures 3 (b) and (c), the 4 new sides of the red pentagon contribute to 5 new edges of the graph. Summarizing, the *graph* of the model comprises the vertices and sides of the pentagons, with the convention that if a vertex lies on the side of another pentagon, then such side will be regarded as two distinct shorter edges. A *hole* is a bounded connected component of the complement of the union of all pentagons.

When a new pentagon is attached, besides the gluing edge, it can touch the existing structure in a combination of 4 different configurations, as shown in Figure 3. In each case, adding a new pentagon increases the number of vertices by at most 3, the number of edges by at most 8 (two per side of the pentagon), and the number of holes by at most 8 (one per new edge). We conjecture the following limits, supported by numerical simulations as shown in Figure 4.



(a) *Edge-edge* **(b)** *Partial edge* **(c)** *Vertex-Edge* **(d)** *Vertex-vertex*

Figure 3: *A new pentagon (red) can touch the rest of the structure in a combination of 4 different ways.*

Conjecture 0.2. *Let n be the number of pentagons in the model, $V(n)$, $E(n)$, and $H(n)$ the number of vertices, edges, and holes, respectively. Then, the expected value of these parameters satisfies the following limits:*

$$\lim_{n\to\infty}\frac{\mathbb{E}[V(n)]}{n}\approx 2.68, \qquad \lim_{n\to\infty}\frac{\mathbb{E}[E(n)]}{n}\approx 4.04, \qquad \lim_{n\to\infty}\frac{\mathbb{E}[H(n)]}{n}\approx 0.36.$$

During the simulations, $V(n)$ and $E(n)$ were kept track of, while $H(n)$ was determined using Euler's formula for planar graphs: $V(n)-E(n)+(n+H(n)+1)=2$. Where the number of faces, $F(n)=n+H(n)+1$, accounts for pentagons, holes, and the unbounded face.

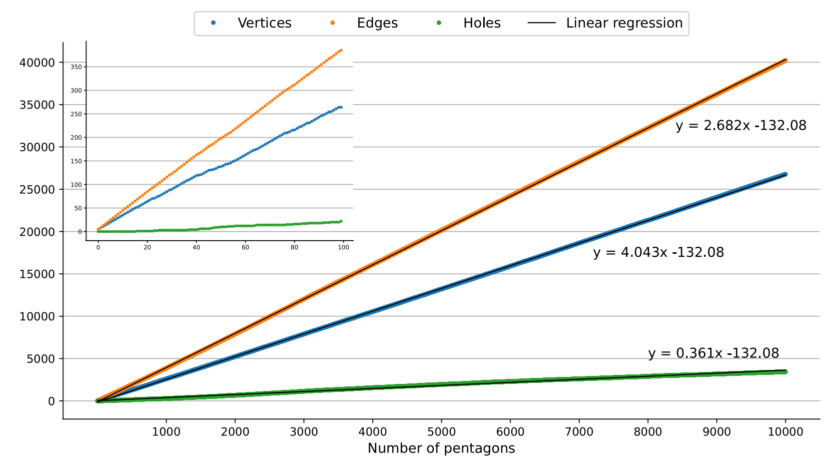


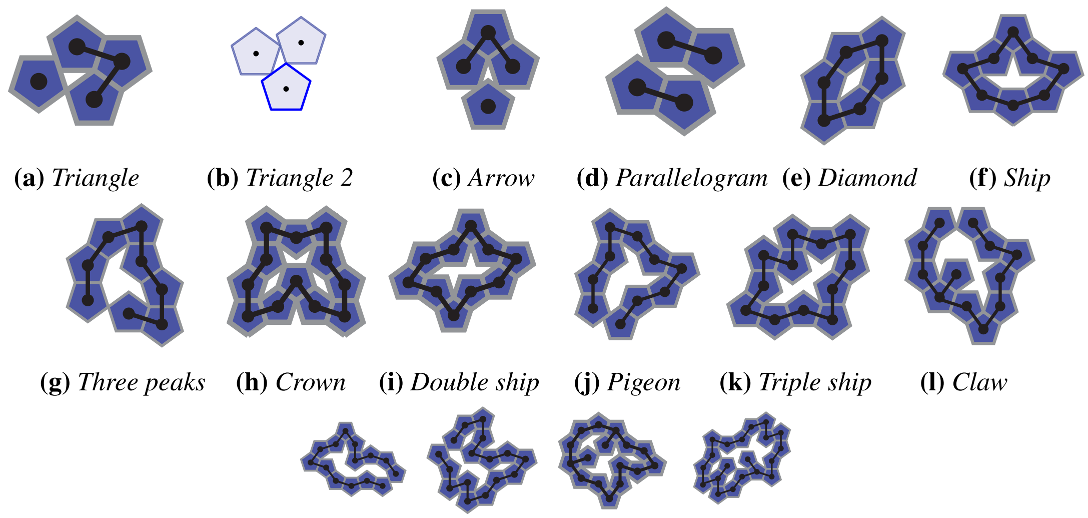
Figure 4: *Growth of graph parameters with respect to the number of pentagons n.*

Description and Analysis of Holes

Recall that all the sides of the pentagons form angles that are multiples of 36° between themselves. Thus, the holes are polygons whose angles are multiples of 36°. Examples of holes discovered in the simulations are illustrated in Figure 5. Consider a hole with l sides whose angles are $36°a_i$ for some positive integers a_i, $1\le i\le l$. The interior angle sum of the hole polygon is $36°a_1+36°a_2+\ldots+36°a_l=180°(l-2)$. This provides a necessary condition for the a_i's, namely, $a_1+a_2+\cdots+a_l=5(l-2)$.

Consequently, up to scale, only two distinct types of triangular holes are possible. These correspond to solutions $(a_1,a_2,a_3)=(1,2,2)$ and $(1,1,3)$, resulting in triangles with angles $(36°,72°,72°)$ and $(36°,36°,108°)$, respectively. Type $(36°,72°,72°)$ is shown in Figure 5(a). We have not observed a hole of type $(36°,36°,108°)$, however it can be created by pentagons whose centers are in the locus of centers of the model, Figure 5(b).

The angle types of quadrilateral holes (up to rotation and reflection) can be $(a_1,a_2,a_3,a_4)=$ (1,1,1,7), (1,1,2,6), (1,2,1,6), (1,1,4,4), (1,4,1,4), (1,2,3,4), (1,2,4,3), (1,3,2,4), (2,2,2,4), (2,4,2,4), (2,2,3,3), or (2,3,2,3). It remains unclear whether these configurations are feasible. Our simulations have shown one hole of type (1,1,1,7) and two instances of type (1,4,1,4), Figure 5 (c), (d), and (e), respectively.



(a) *Triangle* **(b)** *Triangle 2* **(c)** *Arrow* **(d)** *Parallelogram* **(e)** *Diamond* **(f)** *Ship*

(g) *Three peaks* **(h)** *Crown* **(i)** *Double ship* **(j)** *Pigeon* **(k)** *Triple ship* **(l)** *Claw*

(m) *Whale* **(n)** *Fox* **(o)** *Bird* **(p)** *Deer*

Figure 5: *Except for (b), all images of holes were obtained in our simulations. The model is likely to generate infinitely many different holes that remain uncovered forever.*

Towards Developing an Exhibition

Figure 6: *Model with 200 pentagons.*

Figure 6 shows a physical realization of Figure 1(a) using a laser cutter on wood with an engraved acrylic layer lying on top, showing the growing tree. We understand contemporary art through the *working progress* mind setting that gives more value to the artistic process over the final object that is usually prefixed and that reaches an end state. This inspired us to propose a hands-on and collaborative exhibition where the public will place the pentagons on a *citizen simulated* cell growth process on different surfaces at an exhibition room at DESFOGA 2024, Cambados, Spain. DESFOGA is a curatorial program in Cambados that questions powers, inequalities, and all forms of human rights violations through performances, installations, and exhibitions.

References

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