n-Flake Variations

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Abstract

Sierpiński n-gons adapt the idea of Sierpiński's triangle to create fractals from regular n-sided polygons. We develop two variations on this idea. One variation is to fill the center hole with a similar polygon, and the other is to rotate each child polygon a fixed angle from its parent.

Introduction

Sierpiński's fractals have proven popular both mathematically, as their study provides a good introduction to fractal geometry, as well as artistically since their variations produce several esthetically pleasing patterns. Just a few of the types of Sierpiński variations can be seen in the beautiful works of Aceska and O'Brien [1] and several publications by Taylor [7][8]. We will focus on the variations called Sierpiński n-gons [6], also known as n-flakes.

n-Flakes

To construct an n-flake, begin with a regular n-sided polygon. On the inside of each vertex of this polygon, attach a similar polygon. Furthermore, the new polygons at adjacent vertices of the original should just touch with no overlap. The n-flake is the fractal obtained by repeating this motif on each new polygon as shown in Figure 1.

Figure 1: *5-flake (left) and 6-flake (right): original polygon, one iteration, and several iterations of each.*

This construction gives the usual Sierpiński triangle for $n = 3$, but it does not give the standard Sierpiński carpet with $n = 4$. In fact, for the square the result of this process is not even a fractal because there is no hole. See Figure 2.

Figure 2: *3-flake (left) and 4-flake (right): original polygon, one iteration, and several iterations of each.*

Center n-Flakes

The center 5-flake is a well known construction. See modern day mathematician-artists, such as [3] and [5], who have used the 5-flake iteration idea without removing the center pentagon to produce the pentaflake fractal shown in Figure 3. Both of the above two citations credit sixteenth century artist Albrecht Dürer [4] with the idea of tiling the plane with pentagons and rhombuses. For general regular polygons, the hole shape is a similar polygon only when $n = 3$ and $n = 5$. However, one can find a hexaflake with center in many places. Some of the more interesting occurrences are in atennas [2], and in a projection of a hexagonal bipyramid fractal, Hideki Tsuiki's "H fractal," into the plane. He made a movie of several projections [9]. The hexaflake appears at time 4:10 in the video. Both the center pentaflake and center hexaflake are shown in Figure 3.

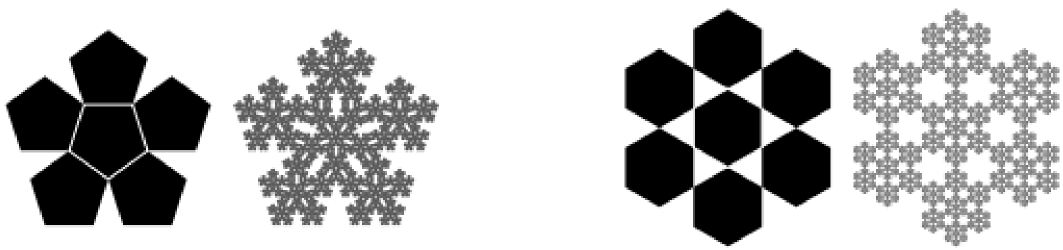

Figure 3: *Pentaflake (left) and hexaflake (right): one iteration and several iterations of each.*

On the other hand, it is harder to find pictures of these center n-flakes for $n > 6$. There are a couple of issues to consider. One is that for pentaflakes and hexaflakes, the center polygon is the same size as the outer polygons, but for $n > 6$, the center polygon is larger than the sibling polygons. Also, it is possible to place the center polygon in different ways. Even more, placement is affected by whether n is even or odd. For n odd, as with the pentaflake, the outer polygons have edges in the direction of the center. A simple placement is to align each edge of the center polygon to each inward facing edge of the outer polygons. For n even, just like with the hexaflake, the outer polygons have vertices pointed towards the center. The simplest placement is to match the center polygon vertices to the vertices of the outer polygons. A 7-flake and an 8-flake are shown in Figure 4.

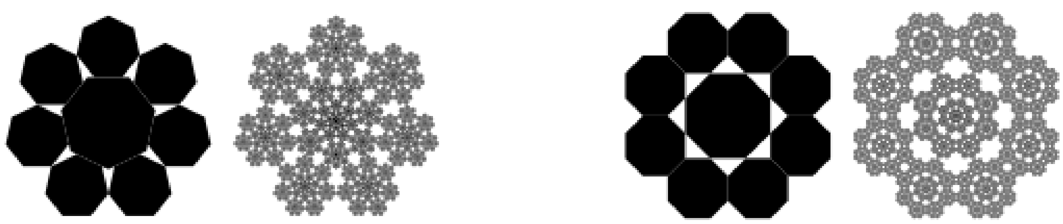

Figure 4: *Septaflake (left) and octaflake (right): one iteration and several iterations of each.*

Rotated n-Flakes

In another variation we rotate each child polygon a fixed angle from its parent polygon. This new variation can be used with either the original n-flake with no center or the center variation. See examples in Figure 5.

- For a regular polygon of n sides, a rotation about the center of $2\pi/n$ radians takes a vertex to the next one. Therefore, all the possible shapes come from angles in the interval $[-\pi/n, \pi/n]$.

- Furthermore, the shape that uses the negative of an angle is the mirror image of the shape for the original angle. One can focus on angles in $[0, \pi/n]$ to see all the essential shapes.

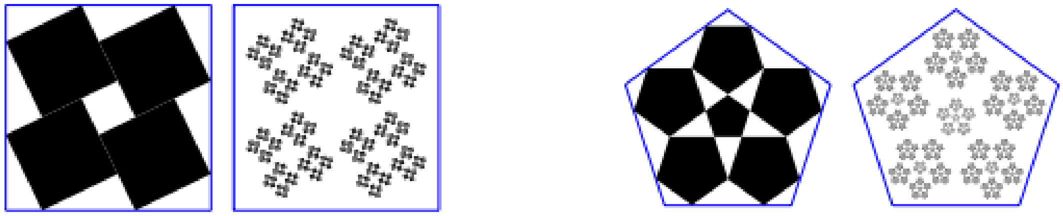

Figure 5: *Rotated n-flakes: one and many iterations with the original polygon outlined in blue. Left: 4-flake no center, angle $\pi/7$. Right: 5-flake with center, angle $\pi/5$.*

Connectivity

The non-rotated fractal has a different aesthetic appeal than its rotated cousins. A large part of this difference has to do with the connectivity properties of these fractals. To simplify this discussion, focus on the original n-flakes that have no centers. For the non-rotated fractal, the resulting object is connected. This is clear since each child shares a vertex with its parent as well as a vertex with each adjacent sibling. These particular vertices then remain in all subsequent descendant polygons so that they are points in the fractal. However, rotated fractals are not connected since child polygons no longer share vertices with either their parents or any of their siblings. See the two-iteration pentaflakes in Figure 6. Parent pentagrams are outlined in blue.

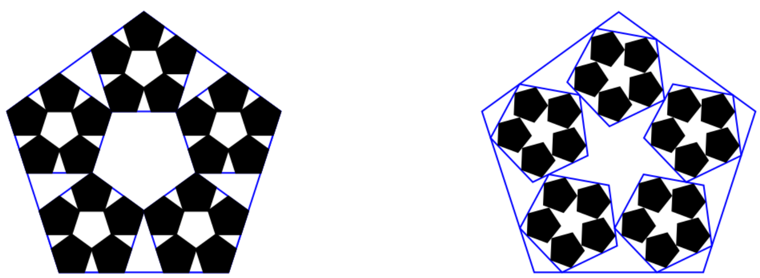

Figure 6: *The left pentaflake is unrotated, while the right one is rotated $\pi/7$.*

Artistic Application

The software we used to generated the center variation n-flakes allows for each middle polygon in the last iteration to be colored differently than the outer polygons. See Figure 7. The second author is an artist who

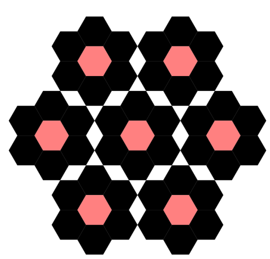

Figure 7: *A center hexaflake with centers colored pink.*

often begins with a patterned background on which he paints. He started with a two-iteration 3-flake with center that is rotated, where he colored the center white, the background color. He printed multiple copies of this 3-flake, interleaved, on a canvas, and then painted this canvas using watered down acrylic paints so that the background pattern is visible. See the 3-flake and painting in Figure 8.

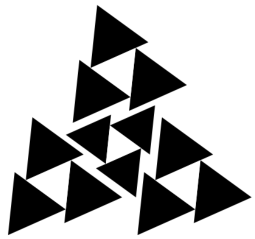

Figure 8: *Left: one rotated 3-flake with white centers. Right: Painting with 3-flake pattern background.*

Acknowledgments

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