Trading Exact for Simple

Wojtek Burczyk[1] and Krystyna Burczyk[2]

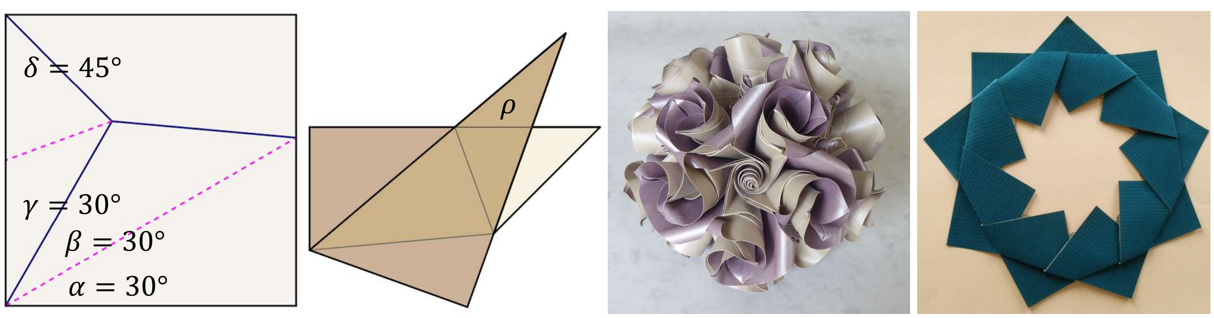
Lisbon, Portugal; [1]woj.burczyk@gmail.com, [2]krystyna@origami.edu.pl

Abstract

Mathematical art typically concentrates on visual representation of an exact solution to a mathematical problem. We present another approach – “trading exact for simple” – that, when applied to origami, opens the door to new creations.

Starting Point

Following a vacation visit to the Palace of Versailles, Krystyna was inspired to design a series of works based on a class of modules inspired by and named after this beautiful place: the Versailles module. Krystyna discovered that the Versailles module could be also arranged into a ring, a rarely-encountered case in origami. In the case of the module shown in the Figure 1 the ring consists of nine modules and incorporates the symmetry of the regular nonagon.



(a) (b) (c) (d)

Figure 1: W*orks made from the same Versailles module (for detailed origami diagrams see[3]): (a) crease pattern of the module, (b) a folded module (c) a polyhedral structure of 60 modules (placed at the vertices of truncated icosahedron) [1], (d) a ring of nine modules [2]*

Both of the above origami works are based on exactly the same module. Its charm comes from a simple structure requiring just four easy folds and from the unusual asymmetry of the folded module. Krystyna also designed four more modules of similar structure that are suitable for rings, and multiple others used to create polyhedral forms. While the Versailles module was conceived with polyhedral form in mind, the ring was a surprise as nothing indicated the rotation angle ρ (an angle between two consecutive modules) would be 40° or $\frac{360°}{9}$. Immediately, the question arose: are there more?

How to Construct the Non-Constructible?

When we first look at the folding sequence of the module (see [3]), we notice the following steps: a construction of a 30° angle (α), a bisector (β, γ), another bisector (δ), a line between two points and a difference of angles (a short valley crease folded when a module is collapsed) and several reflections. All of these are compass-and-ruler constructions; however, construction of a regular nonagon (or construction of the 40° angle) with a compass and ruler is impossible. Therefore it was a surprise to find nine modules assembling into the regular nonagonal structure of Figure 1(d). How is this ring constructible if it is not?

When we compute the rotation angle ρ (see Figure 2(a)) on the base of the folding process, it gives approximately $\rho = 40.2°$. The difference between the ideal nonagon and the folded model is about 0.2° per module, or 1.8° for a whole ring. Such a small difference is lost in the natural imperfection of folding.

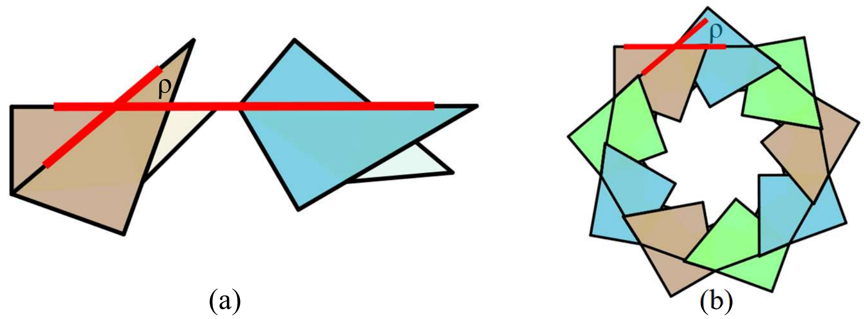


Figure 2: *The rotation angle ρ is determined by the shape of a module: (a) two modules ready to be joined, red marked edges determine the rotation angle, (b) the rotation angle in a ring*

Sesame, Open Up!

It seemed apparent that there should be more nice rings. First Krystyna set out to find few of them. Second, if we look for the crease pattern of a module (Figure 1(a)) we may see three degrees of freedom (angles α, γ, δ) and only one constraint for the rotation angle $\rho = \frac{360°}{n}$, where n is the number of modules in a ring.

We may write an equation for ρ with α, γ, δ as variables and solve the equation $\rho(\alpha, \gamma, \delta) = \frac{360°}{n}$ for the desired rank of a ring n. For instance, if we select one parameter model with α as a single parameter (see Figure 3) the solution for a pentagonal ring is $\alpha = 14.0865°$ ($tan(\alpha) = 0.245856$). But this solution is completely useless. How can we fold such an angle? If we try to approximate the angle we will finish with a long and messy sequence of folds. We hit a dead end: even if we calculate something, we will lose all the charm of the original model trying to construct the solution.

The answer comes from the geometry of the Versailles ring shown in the Figure 1(d).

The Problem Attacked from Behind

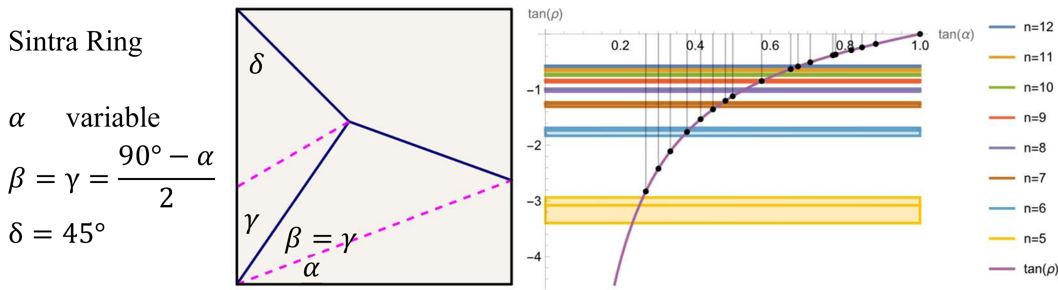
Since approaching this head-on (starting from an exact solution for ρ) won't work, we must cheat and attack from behind. We will trade **precise** (mathematically correct) construction for **easy to fold** and **precise-enough** construction. Our original problem now converts into a new one: find an easy construction of angles α, γ, δ that gives precise-enough rotation angle $\rho \approx \frac{360°}{n}$.

First we must define what is *easy* and *precise enough*. One of the easiest origami constructions is *fold a line onto a line*, or construction of the bisector of the angle between two lines. Another easy construction is to fold a line coming from a corner of a sheet of paper such that another corner lands on an existing line (such a fold is frequently used for construction of 30° angle, the existing line is a vertical line at the midpoint of the top edge of the sheet). We will use our construction for a vertical lines that divide the edge into m intervals of equal length, where $m = 2,4,8,3,5,6$. Such divisions are easy to obtain by standard origami constructions. A combination of such easy elementary constructions gives us an easy construction for angles. We also want to limit ourselves to angles of 15° or greater, as small angles are more difficult to fold. We will denote the set of angles of interest by EA.

What is *precise enough* is a matter of experiment. Actual folding showed that it is easier to close a ring if the rotation angle ρ is in excess of $\frac{360°}{n}$, as loose folding allows for slight variations of the angle between the modules. If the rotation angle ρ is less, joining the modules is more difficult as they must be stretched a bit to form the ring. The *precise enough* range of the angle was set experimentally to no more than 8° and no less than 4° (total divergence for all modules in a ring). We will denote the interval for a specific rank of ring by $PER(n) = \left[\frac{360°}{n} - 4°, \frac{360°}{n} + 8°\right]$ and the union of intervals for all ranks by PER.

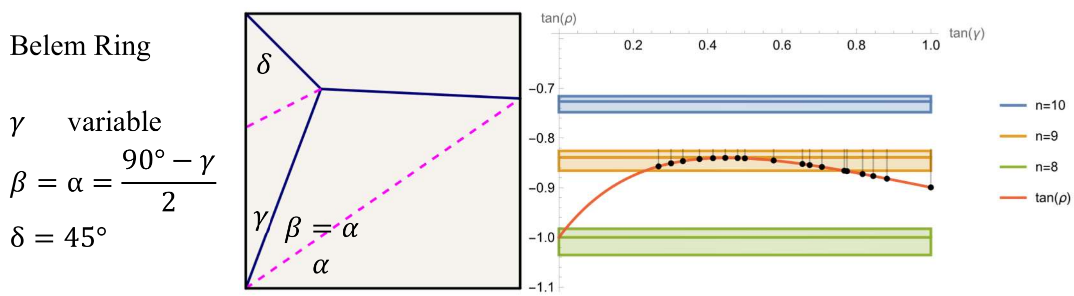
Now, we are armed to attack our problem. We will solve $\rho(\alpha,\gamma,\delta) \in PER$ for $\alpha,\gamma,\delta \in EA$. As the set EA has limited size, we do not need any advanced algorithm, we just check all possible triples for the condition above. The selected triples are tested for additional conditions that guarantee the resulting folds will form a ring-compatible module. We considered four models with one to three degrees of freedom. Following the name coming from the palace of Versailles we gave names to the models and resulting rings after royal palaces around Lisbon, where we currently live.

Figure 3 presents results for a 1-parameter model (for Sintra Rings), where α is the only variable, γ (and β) is the bisection of the angle complementary to α, and $\delta = 45°$ (bisection of the right angle). Table 1 below shows some statistics for screening the space of solutions.



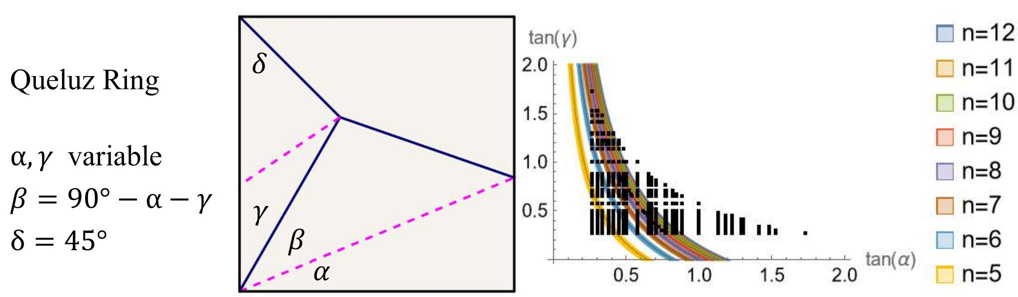
(a) (b)

Figure 3: *Sintra Rings generated by a model with one variable α: (a) the crease pattern for a module, (b) the solution, the curve shows the rotation angle as a function of α, the dots represent rotation angles for easy angles, the color strips represent the precision enough range for a particular ranks of a ring (a color line represents an exact rotation angle)*



(a) (b)

Figure 4: *Belem Rings generated by a model with one variable γ: (a) the crease pattern for a module, (b) the solution, the curve shows the rotation angle as a function of γ, the dots represent rotation angles for easy angles, the color strips represent the precision enough range for particular ranks of a ring (a color line represents an exact rotation angle)*



(a) (b)

Figure 5: *Queluz Rings generated by a model with two variables α, γ: (a) the crease pattern for a module, (b) the solution, a curve shows the variables α, γ giving the exact rotation angle for a specific rank of a ring, a color strip represent the precision enough range, the dots represent pairs of easy angles*

Another 1-parameter model (for Belem Rings) may be started from the angle γ. The angles α (and β) are the bisections of the angle complementary to γ, and $\delta = 45°$. Figure 5 shows a crease pattern for a module and solution of the problem in graphical form. Table 1 below shows some statistics for screening the space of solutions. We were surprised by how differently the Belem Rings behaved as compared to Sintra Rings – easily visible if we compare graphs of solutions. Notable is that all solutions of $\rho(\gamma) \in PER$ for $\gamma \in EA$ (approximate solutions) generate 9-pointed rings, but there is no exact solution $\rho(\gamma) = 360°/9$ for a 9-pointed ring. This means that without straying from the traditional approach (find an exact solution then fold it exactly or approximately) all these rings would have remained undiscovered!

Queluz Rings are generated by a 2-parameter model. The angles α and γ are independently variable. The angle β is the remaining part of the right angle and $\delta = 45°$. Figure 5 shows a crease pattern for a module and solution of the problem in graphical form. Table 1 below shows some statistics for screening the space of solutions.

Mafra Rings are generated by a 3-parameter model. The angles α, γ, δ are independently variable. Table 1 shows some statistics for screening the space of solutions.

Table 1: *Number of Origami Designs Generated.*

	Sintra Rings	Belem Rings	Queluz Rings	Mafra Rings
Easy Angles	35	35	399	12320
Easy Angles in Enough Precision Range	4	14	94	1808
Rejected because of Origami Constraints	0	2	42	1415
Valid Designs to Be Folded in Real Paper	4	12	52	393
Yield	11,4%	34,3%	13,0%	3,2%

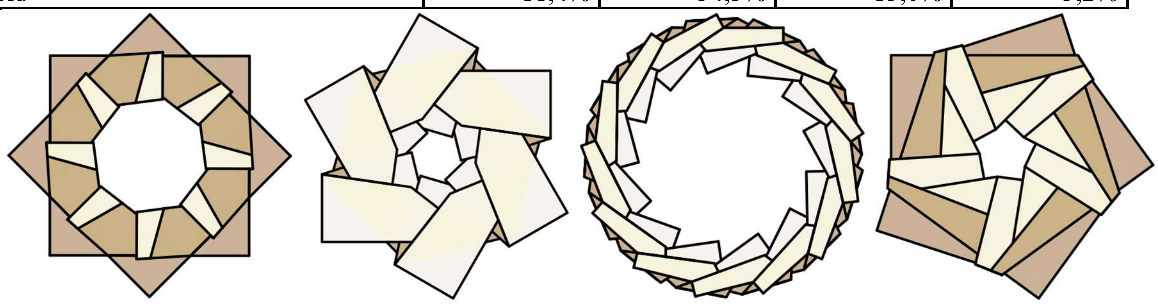


Figure 6: *Examples of Generated Rings*

Summary and Conclusions

During the research we were surprised several times. First, we expected a dozen designs and got hundreds. Second, not only are all ranks of rings from 5 to 20 represented (with the exception of 18), but also many additional rings of unusual ranks (7, 9, 11, 13) rarely seen in origami designs. Third, the generated designs show a wide diversity of visual effect, many diverging far from the design we started with (compare Figure 6 and 1). And finally, it works! The designs are not only possible to fold with physical paper but, as we postulated, are easy to fold!

The concept of trading exact for easy has proven its value not only as a technical trick, but also as a general guideline. By not limiting ourselves to an exact solution, it is possible to discover previously-unreachable designs and new opportunities for creativity.

References

[1] K. Burczyk. *Versailles: King (K21027),* 2021, paper, private collection.

[2] K. Burczyk: *Versailles Ring of 9 modules (30/30/30) (I10003)*, 2010, paper, private collection.

[3] K. Burczyk, W. Burczyk. "Once Upon a Time in Versailles." *The Fold*, no. 66, 2021. https://origamiusa.org/thefold/article/once-upon-time-versailles-part-1 and an appendix https://origamiusa.org/system/files/thefold/thefold65_kusudama_D143_burczyk.pdf.