Volume-Enclosing Minimal Surfaces of Torus Knots and Links

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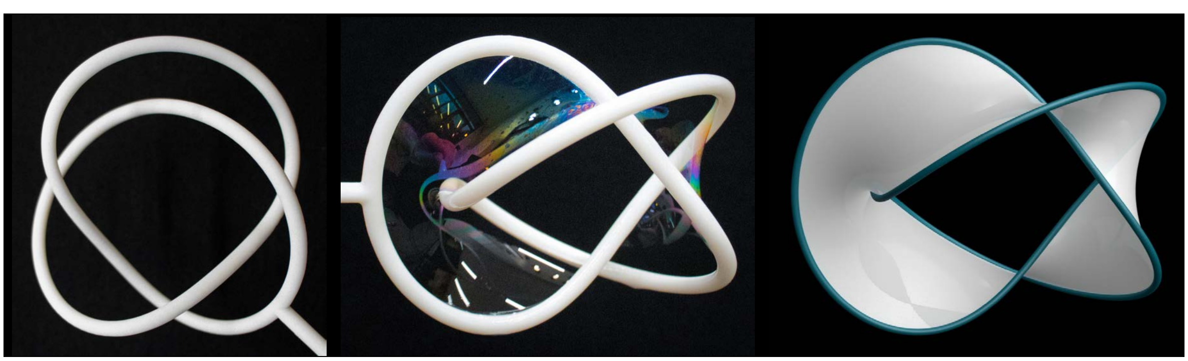
Abstract

Certain surfaces created by the enclosures of various torus curves are popular mathematical icons, most notably including the Möbius strip and art of twisted tori. Minimal surfaces are also popular due to typically smooth curvature and can be combined with torus curves; a minimal surface formed on a (2,1) torus knot is a smooth Möbius strip. By considering the minimal surfaces of (3,1) and higher torus curves, versions of twisted tori are created that enclose volumes with cross sections of concave polygons. Torus knot minimal surfaces are interesting as they contain one outer surface; the (4,1) torus knot is reminiscent of a Penrose triangle with rounded corners. Since the multiply symmetric volume is enclosed by a smooth outer surface, the shape has a modern aesthetic potentially useful in jewelry and architecture and may also have applications in material science and engineering as a minimal surface.

Minimal Surfaces, Knots, and Links

Minimal surfaces have been studied for their intriguing properties and have a longstanding place in mathematical iconography. A *minimal surface* is a surface that locally minimizes its surface area [5]; this means in any small region, a deformation would increase the surface area. Meusnier showed that this implies that the minimal surface has zero mean curvature. Some early minimal surfaces include planes, catenoids and helicoids. Minimal surfaces also reach minimal surface tension energy, a property useful to practical implementations. An *area-minimizing* surface is a unique minimal surface that has the global minimum surface area for a given boundary; soap films naturally form area-minimizing surfaces. Minimal surfaces have been absorbed into wider culture including sculptures (such as [3]) and depictions of minimal surfaces.

Knots, links, and related surfaces have also occupied the minds of mathematicians and artists [6]. A *mathematical knot* is an embedding of the circle into three-dimensional space. In comparison, *links* are similar to knots, but are two or more interlinked closed curves. An everyday knot, as you might tie your shoelaces, can be transformed into a mathematical knot by connecting the two free ends to form a closed curve. For the “overhand” knot, a closure of the free ends creates the Trefoil knot. The minimal surface of a Trefoil knot creates the three-twist Möbius strip as shown in Figure 1, another art icon composed of a single, looped surface with loop ends twisted three 180-degree twists from each other. Interest in knots, links, and their relation to surfaces is ongoing, including study of Seifert surfaces.



(a) *A Trefoil knot wireframe* **(b)** *Soapfilm on wireframe* **(c)** *Minimal three-twist Möbius strip*

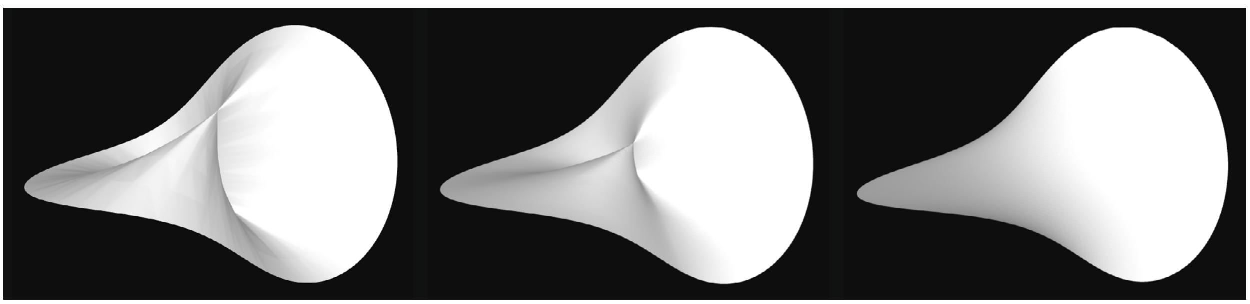
Figure 1: *Construction of a three-twist Möbius strip from the soapfilm of a Trefoil knot*

Torus Curve Minimal Surfaces

The Trefoil knot is an example of a *torus curve*, curves which can be placed on the surface of a torus. A parametric equation for these curves can be found in Equation 1, where p and q are definable parameters, for example the Trefoil knot is a (2,3) torus knot where $2 = p$ and $3 = q$.

$$\begin{aligned} x &= \cos p\phi(r_o + r_i \cos q\phi), && \phi \in [0, 2\pi) \\ y &= \sin p\phi(r_o + r_i \cos q\phi), && r_i, r_o = \text{the inner and outer radius of the torus} \\ z &= h_a r_i \sin q\phi && h_a = \text{optional height multiplier for aesthetic look} \end{aligned} \tag{1}$$

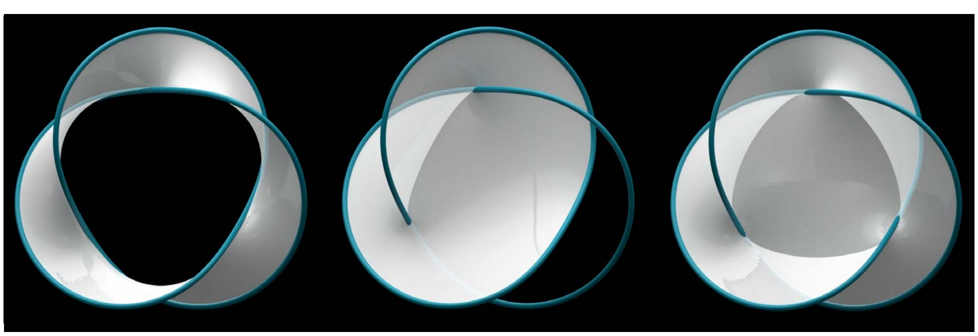
Setting $p = 1$, $q = n$, for $n > 1$ a surface similar to Plücker's nth conoid is created except with a boundary given by the parametric equation in Equation 1. However, these initial surfaces are not minimal and a soapfilm on this curve would then be unstable. Using the Surface Evolver software [1], a surface mesh can be defined for this surface as in Figure 2a. By reducing the simulated surface tension energy of the surface mesh with the conjugate gradient method, the surface evolves to a stable saddle surface with minimal mean curvature as seen in Figure 2c.



(a) *(1,2) torus surface* **(b)** *Surface area begins reducing* **(c)** *Minimal surface for (1,2) curve*

Figure 2: *The original construction of a (1,2) torus surface is unstable and reduces to a minimal surface*

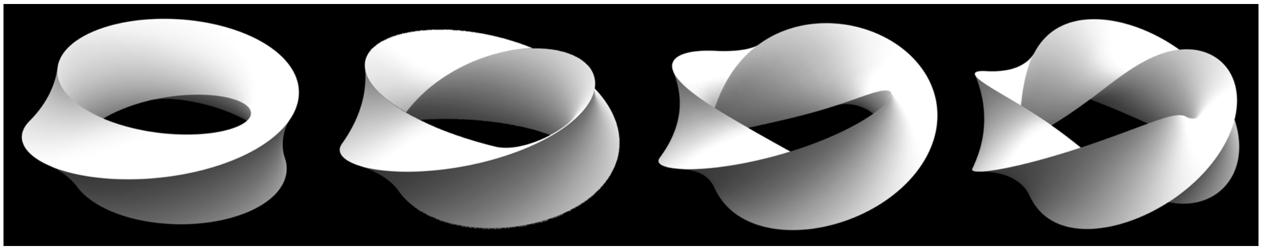
Shapes found from Equation 1 when $p = 2$ and q is odd produce knots, surfaces found on these knots include n-twist Möbius strips. When n is even, the curve is degenerate, but can be rotated to form a link. When $n = 2$, the Hopf link is found and when $n = 4$, this forms Solomon's knot (a misnomer since it is a link). When $n = 3$, the shape is a three-twist Möbius strip. Dissimilar minimal surfaces on the same curve are possible, as seen in Figure 3 [1], each of which is stable with a physical soap film. This paper considers complex geometries where $3 \leq p \leq 5$ and $1 \leq q \leq 4$; because multiple minimal surfaces may exist, this paper considers minimal surfaces that are constrained to the entire curve and enclose volumes.



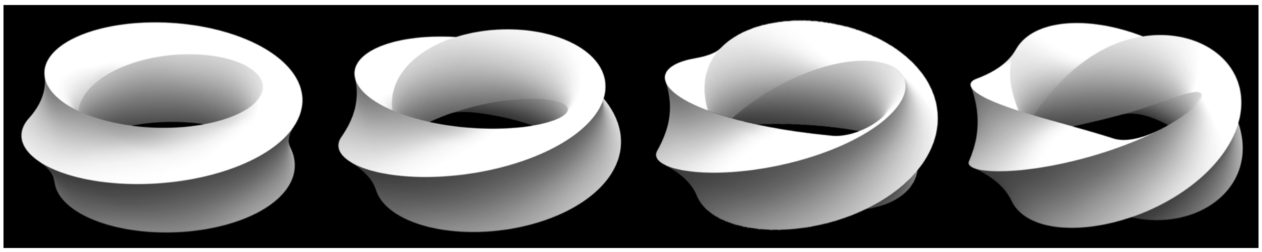
(a) *3-twist Möbius strip* **(b)** *Alternate stable surface* **(c)** *Additional stable surface*

Figure 3: *The stable soap films of the Trefoil knot*

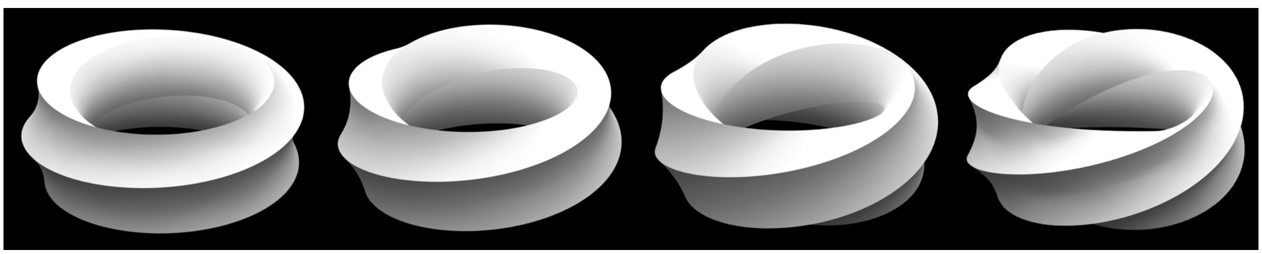
Simulated Soapfilms of Torus Curves, Knots, and Links



(a) *(3,1) torus knot surface* **(b)** *(3,2) torus knot surface* **(c)** *(3,3) torus link surface* **(d)** *(3,4) torus knot surface*



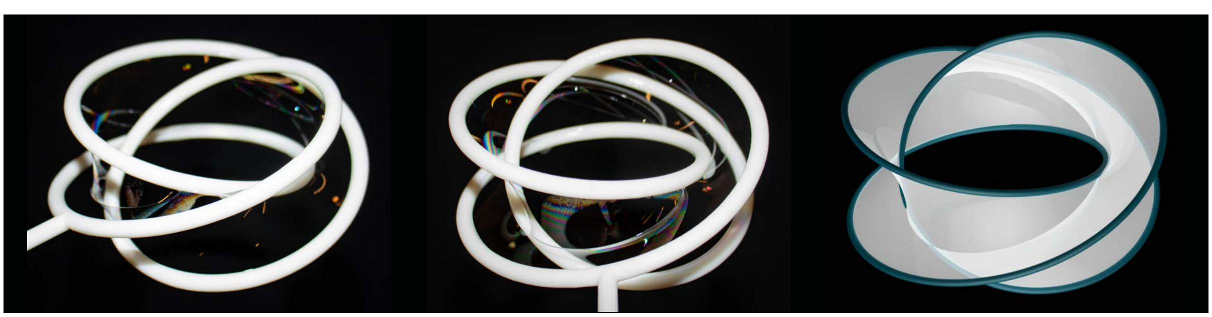
(e) *(4,1) torus knot surface* **(f)** *(4,2) torus link surface* **(g)** *(4,3) torus knot surface* **(h)** *(4,4) torus link surface*



(i) *(5,1) torus knot surface* **(j)** *(5,2) torus knot surface* **(k)** *(5,3) torus knot surface* **(l)** *(5,4) torus knot surface*

Figure 4: *Minimal surfaces bound by torus curves where* $r_i = 0.5$*,* $r_o = 2$*, and* $h_a = 2$

By finding points on torus curves according to Equation 1 and reducing the surface tension energy through iteration with Surface Evolver [1], minimal surfaces formed with boundaries on torus curves from (3,1) to (5,4) are found as seen in Figure 4. For torus curves where the greatest common factor of p and q is 1, the volume is created from a single surface. Cross sections of these volumes have p vertices with concave edges. Dipping a wireframe in a soap solution does not produce these surfaces—instead making zero-volume soap films such as those found in Figure 5—as soap films form the unique area-minimizing surface. Nevertheless, calculations with Surface Evolver indicate that the surfaces in Figure 4 are minimal.



(a) *Soap film on wireframe* **(b)** *Oriented soap film on wireframe* **(c)** *Simulated soap film surface*

Figure 5: *(3,2) torus knot minimal surface soap films show an alternate minimal surface*

The minimal surface is represented by a meshing that can be modified for various aesthetic looks and 3D printing purposes. For example, the curve as defined by Equation 1 can be thickened and attached to a wand handle to create a wireframe as seen in Figure 6a and throughout. Additionally, the shape may be meshed in a way that would create a surface texturing as seen in the triangular pattern in Figure 6b or hexagonal pattern in Figure 6c. These representations bring a more artistic approach to abstract mathematical surfaces.

(a) *Curve thickening* **(b)** *Mesh edge tubing* **(c)** *Facet center tubing*

Figure 6: *3D-printed (3,2) torus minimal surface with different mesh postprocessing effects*

Summary and Conclusions

The surfaces found from enclosing torus knots and links have been shown in various representations. These shapes possess a curious nature, similar to the Penrose triangle or Möbius strip. Because they enclose volumes, these shapes are well-suited for applications in sculpting, architecture, or jewelry, as demonstrated by the necklace pendant 3D printed by the author in rose gold, shown in Figure 7. Additionally, because the shapes are minimal surfaces, they may have application to materials science or engineering applications.

Figure 7: *Usage in jewelry*

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