Hyperbolic Isogonal Tilings from Uniform Edge Colorings

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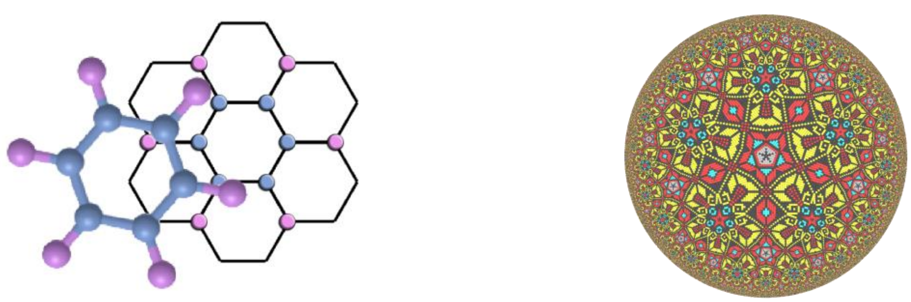
Abstract

This paper presents the construction of an isogonal tiling from an edge coloring of a uniform tiling of the hyperbolic plane. In addition, it also discusses how the isogonal tiling can be transformed into a hyperbolic artwork by using motifs with appropriate symmetry properties as the edges of the tiling.

Introduction

In tiling theory, the study of tilings with vertex transitivity properties has always been of interest (see [3][5][8] and references herein). These tilings have been used for their applications in science, such as to model various chemical structures. For example, a patch of a two-coloring of the hexagonal tiling in Figure 1(a) can be used to model benzene [3]. In art, these tilings may also be used as a basis for constructing aesthetically pleasing colored patterns. The regular (5^5) tiling of the hyperbolic plane ($\mathbb{H}^2$) with symmetry group $*552$ for instance, was used to render a hyperbolic pattern presented in Figure 1(b) using as motifs the textile designs from the Yakans, one of the indigenous communities in the Philippines [6].



(a) (b)

Figure 1: *(a) A patch of a two-coloring of a hexagonal tiling used to model benzene; and (b) a hyperbolic pattern with motifs inspired by the indigenous designs of the Yakan tribe from Mindanao, Philippines.*

In this paper, we discuss the construction of isogonal tilings of $\mathbb{H}^2$. *Isogonal tilings* are tilings with vertices forming one orbit under the action of their respective symmetry groups [9]. This means that in an isogonal tiling, there is always a symmetry of the tiling that will send one vertex to another. In [8], a list of Euclidean isogonal tilings was derived using the concept of edge adjacency symbols. There is existing literature on hyperbolic uniform tilings [5][7] which are isogonal tilings consisting solely of regular polygons, but not much has been said on a systematic construction of isogonal tilings of $\mathbb{H}^2$ in general. In this work, an approach to construct an isogonal tiling of $\mathbb{H}^2$ from a uniform tiling will be presented, by considering its uniform edge coloring. It can be recalled that an edge coloring of a uniform tiling is *uniform* if for any two vertices of the tiling, there is a symmetry of the tiling that sends one vertex to the other and preserves the colors of the edges [9].

Constructing Isogonal Tilings from Uniform Edge Colorings

We begin by first considering a uniform edge coloring of a uniform tiling. Let $\mathcal{T}$ be an uncolored uniform tiling with symmetry group G. Choose a subgroup H of G such that H forms one orbit of vertices of $\mathcal{T}$. Suppose H forms n orbits of edges in $\mathcal{T}$, that is, if E is the set consisting of edges of $\mathcal{T}$, E can be partitioned as $E = He_1 \cup He_2 \cup \cdots \cup He_n$ where we pick $e_1, e_2, \dots e_n$ as representatives from each orbit of edges. Assigning distinct colors to each He_j, $j = 1, \dots n$, we obtain an edge-n-coloring of $\mathcal{T}$. Since H preserves the colors of the edges, the edge-n-coloring is uniform.

Now, to construct an isogonal tiling $\mathcal{T}^*$ from the uniform edge coloring of $\mathcal{T}$, a new set of edges $e_1^*, e_2^*, \cdots, e_n^*$ is introduced to replace $e_1, e_2, \dots, e_n$, respectively. There are four possibilities for e_j^* $(j = 1, \dots n)$ depending on its finite group F of symmetries in the group G. The group F is either of type C_n (cyclic group of order n) or D_n (dihedral group of order $2n$). Figure 2(a)-2(d) shows the four possibilities for e_j^* with various symmetry types. Figure 2(a) type C_1: F is generated by the identity isometry, Figure 2(b) type C_2: F is generated by a 180° rotation with center C, Figure 2(c) type D_1: generated by a reflection with axis l, and Figure 2(d) type D_2: generated by two reflections with axes perpendicular to each other, one of which passes through the edge. It is important to note that we can replace the edge e_j with one of the four possibilities of e_j^*, provided the symmetries of e_j in H is contained in F.

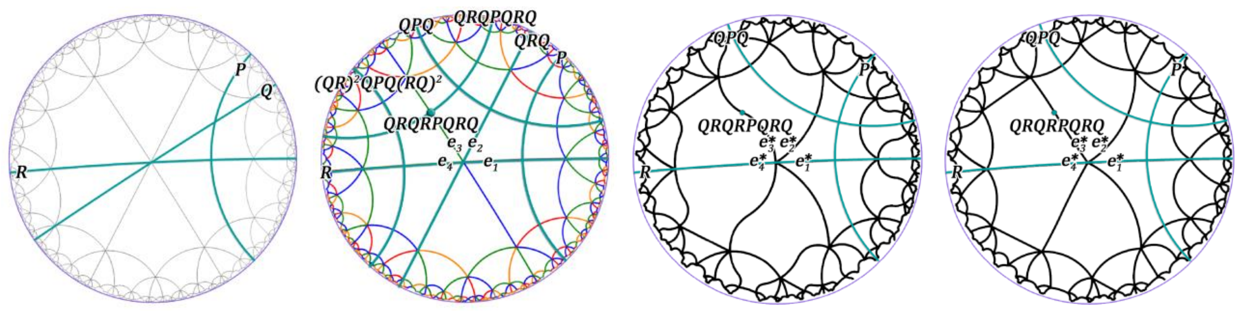
Figure 2: *Edges with finite symmetry group types (a) C_1; (b) C_2; (c) D_1; and (d) D_2.*

The tiling $\mathcal{T}^*$ is formed by applying H to the newly constructed edges $e_1^*, e_2^*, \cdots, e_n^*$ .That is, $\mathcal{T}^* = He_1^* \cup He_2^* \cup \cdots \cup He_n^*$. The tiling $\mathcal{T}^*$ is a tiling whose vertices are the vertices of $\mathcal{T}$, the edges are the union of orbits of the e_j^*'s under H and the tiles are the regions bounded by these edges. Its symmetry group is H. Since H is transitive on the vertices of $\mathcal{T}$, it follows that $\mathcal{T}^*$ is an isogonal tiling.

To illustrate the construction, consider the uniform tiling $\mathcal{T} \coloneqq (4^6)$ of $\mathbb{H}^2$, a tiling with 6 regular 4-gons incident to each vertex. Its symmetry group is $G = \langle P, Q, R \rangle \cong *642$ generated by reflections P, Q, and R with axes shown in Figure 3(a). We consider a subgroup $H = \langle P, R, QPQ, QRQRPQRQ \rangle \cong 2^*222$ of G generated by reflections P, R, and QPQ, and the 180° rotation $QRQRPQRQ$ with axes and center, respectively, shown in Figure 3(b), where H forms one orbit of vertices of $\mathcal{T}$. Moreover, H forms four orbits of edges of $\mathcal{T}$ namely He_1, He_2, He_3 and He_4. Assigning the colors red, blue, green, and orange, respectively to He_1, He_2, He_3 and He_4, we obtain the uniform edge coloring in Figure 3(b).

We now construct an isogonal tiling from this uniform edge coloring of $\mathcal{T}$. To do this, we introduce a new set of edges $e_1^*, e_2^*, \dots, e_4^*$. We explain the construction in detail. We use the straight edge e_1^* of symmetry group in G generated by reflections P and R. It can be checked that the symmetry group of e_1 in H is also generated by P and R. The edge e_2^* may be an edge of symmetry group in G generated by the reflection QPQ or generated by perpendicular reflections QPQ and QRQ. Note that the symmetry group of e_2 in H is generated by QPQ. We choose e_2^* of symmetry group generated by QPQ. The edge e_3^* may be an edge of symmetry group in G generated by the 180° rotation $QRQRPQRQ$ or generated by the 180° rotation $QRQRPQRQ$ and reflection $QRQPQRQ$. Observe that the symmetry group of e_3 in H is generated by $QRQRPQRQ$. We choose e_3^* of symmetry group generated by $QRQRPQRQ$. Lastly, we use a straight edge e_4^* of symmetry group in G generated by perpendicular reflections R and $(QR)^2QPQ(RQ)^2$. Observe that the symmetry group of e_4 in H is also generated by R and $(QR)^2QPQ(RQ)^2$. We now form the isogonal tiling $\mathcal{T}_1^*$ by forming the union of the orbits of these new edges under the subgroup H. That is, $\mathcal{T}_1^* = He_1^* \cup He_2^* \cup He_3^* \cup He_4^*$ (see Figure 3(c)).

Another isogonal tiling may be formed by choosing a different set of edges for e_1^*, e_2^*, e_3^*, and e_4^*. The isogonal tiling $\mathcal{T}_2^*$ in Figure 3(d) is formed by considering $He_1^* \cup He_2^* \cup He_3^* \cup He_4^*$ where the symmetry group of e_1^* in G is generated by the reflections P and R, the symmetry group of e_2^* in G is generated by the reflection QPQ, the symmetry group of e_3^* in G is generated by the $180°$ rotation $QRQRPQRQ$ and reflection $QRQPQRQ$, and the symmetry group of e_4^* in G is generated by the perpendicular reflections R and $(QR)^2QPQ(RQ)^2$.



(a) (b) (c) (d)

Figure 3: *(a) The hyperbolic tiling $\mathcal{T}$ with symmetry group G generated by reflections P, Q, R; (b) a uniform edge-4-coloring of $\mathcal{T}$, generators $P, R, QPQ, QRQRPQRQ$ of H and axes of reflections QRQ, $QRQPQRQ$, $(QR)^2QPQ(RQ)^2$; and (c-d) isogonal tilings $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$ with symmetry group H.*

Constructing Patterns from Isogonal Tilings

An isogonal tiling may be transformed to arrive at a pattern with varying motifs, having the same symmetry properties as the tiling. To illustrate this, consider the hyperbolic isogonal tiling $\mathcal{T}_1^*$ shown in Figure 3(c). We consider the symmetry group in G of each edge $e_j^*, j = 1, \dots, 4$ in $\mathcal{T}_1^*$ and construct a motif with the same symmetry group in G as that of e_j^*. For example, the edge e_1^* of symmetry group in G generated by two reflections P and R is replaced by a blue butterfly of symmetry group in G also generated by P and R. The rest of the edges are replaced as follows: The edge e_2^* of symmetry group in G generated by the reflection QPQ by a green dragonfly of symmetry group in G generated by the reflection QPQ; the edge e_3^* of symmetry group in G generated by the $180°$ rotation $QRQRPQRQ$ by a yellow flower of symmetry group generated by $QRQRPQRQ$; and the edge e_4^* of symmetry group in G generated by perpendicular reflections R and $(QR)^2QPQ(RQ)^2$ by the pink butterfly of symmetry group in G generated by perpendicular reflections R and $(QR)^2QPQ(RQ)^2$. The resulting pattern is shown in Figure 4(a) with symmetry group 2^*222 and having four different classes of motifs corresponding to the four orbits of edges in the original tiling $\mathcal{T}_1^*$.

Similarly, Figure 4(b) and Figure 4(c) show patterns corresponding to the isogonal tiling $\mathcal{T}_2^*$. The motifs used in Figure 4(b) are inspired by the fabric designs of the Northern Kankana-ey from Northern Luzon in the Philippines [2]. The motifs used to replace e_1^*, of symmetry group in G generated by two reflections P and R, and e_3^* of symmetry group in G generated by the $180°$ rotation $QRQRPQRQ$ and reflection $(QR)^3Q$, are known as *matmata* which represents rice grains and the eyes, as they admire rice as an all-seeing god that gives their body the nourishment that it needs. The motifs used to replace e_2^*, of symmetry group in G generated by the reflection QPQ, and e_4^* of symmetry group in G generated by the perpendicular reflections R and $(QR)^2QPQ(RQ)^2$, are called *tiktiko*, which shows distinguishing zigzag designs that symbolize the mountains and forests where their rice fields are located. These two types of motifs imply wealth and abundance for the Northern Kankana-ey.

Moreover, Figure 4(c) shows a hyperbolic pattern with motifs that are inspired by the indigenous designs of the Yakan tribe in the Philippines [4]. The designs of the Yakans usually include colorful motifs to showcase bravery in battle, joy in birth, and marriage rituals. The motifs used to replace e_1^*, e_2^*, e_3^* and e_4^*

consist of colorful diamonds which is a traditional motif used in Yakan textiles. These motifs also show symmetries which satisfy the symmetry group condition mentioned in the previous section.

(a) (b) (c)

Figure 4: *Hyperbolic isogonal tiling using different motifs as edges: (a) flowers and butterflies; motifs from indigenous designs of the (b) Northern Kankana-ey; and (c) Yakan tribe in the Philippines.*

Conclusion

The connection of uniform edge colorings and isogonal tilings may lead to the construction of other types of tilings such as k-isogonal tilings from edge colorings of k-uniform tilings of the Euclidean plane, hyperbolic plane and 2-sphere. The problem of how to efficiently construct uniform edge-n-colorings has been addressed in response to the problem posed by Grünbaum and Shephard in [9] and is discussed in detail in [1]. Consequently, various aesthetically pleasing patterns may also arise depending on the underlying isogonal or k-isogonal tiling and the set of motifs that will be used.

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