A Group Activity to Make a Hat Tiling with Paper

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Abstract

This article gives step-by-step instructions for a group activity in which participants with no mathematical training assemble paper copies of the "einstein" hat tile to create a large patch of the aperiodic hat tiling.

Introduction

We see examples of tilings everywhere—on sidewalks, on the floors of kitchens and bathrooms, and as decorative designs on walls. These designs are often highly symmetric. However, we have known for some time that it is possible to find a small number of tiles that can cover the entire plane but only in a way without periodicity (see [2, Ch. 10], for instance). Such tilings are called aperiodic.

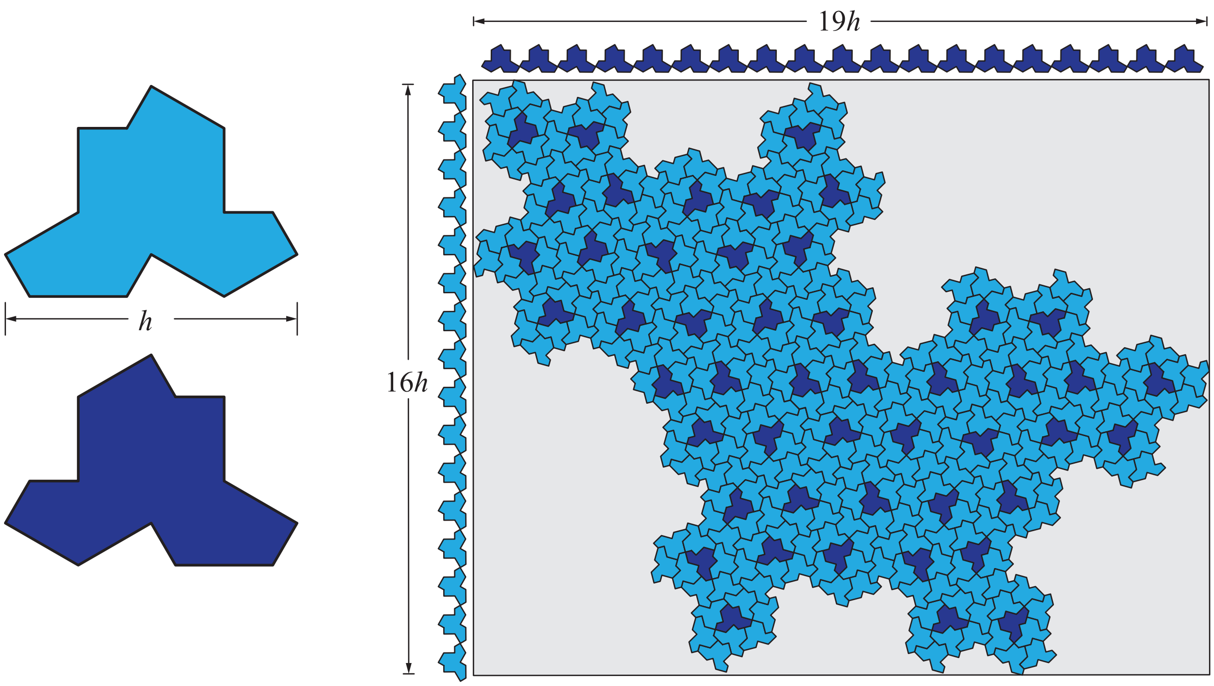

Figure 1: *This tiling is made from hat tiles (light blue) and their reflections (dark blue). The space required to make this patch of the hat tiling is at least* 19×16 *measured in hat-tile widths.*

In this article, we describe a group activity that explores the fascinating world of aperiodic tilings using the "einstein" hat tile discovered by David Smith in December 2022 and announced to the world in March

2023 [3]. The hat tile is a single shape that can tile the plane but only nonperiodically. The activity has been run at Gather 4 Gardner 15 and with a group of undergraduate mathematics majors (see Figure 2).

The procedure is based on one of the substitution methods shown in [3, Figure 2.11] and could, at least in principle, be extended to larger tilings. See, also, [1].

All you need for this activity, which is designed for participants with any level of mathematical expertise, is a generous open area, colored paper, a printer, scissors, clear tape, painter's tape (or masking tape), and some patience. The goal of the activity is to assemble the hat tiles into a patch like those in Figures 1 and 2.

Instructions

1. Find a large space. It is important to find enough floor space to accommodate the tiling. As shown in Figure 1, the space required for the build is at least $19h$ by $16h$ where h is the width of your hat tile.

2. Print and cut the paper tiles. The activity requires 392 hat tiles. Of these, 343 will have one orientation, and 49 will be reflected copies of those. The numbers of each type can be ignored during the preparation phase if there is no chosen color scheme for the eventual tiling (since any paper tile can be flipped over). However, if you wish to make the reflected copies a different color than the rest, then this is an important detail.[1] In these instructions, the dark blue tiles are reflected copies of the light blue tiles. Printable sheets with various sizes of hat tiles are available as a supplement on the Bridges webpage for this article. Print enough copies to make the tiling and cut them out. They can be cut out with scissors; however, a paper cutter like a Cricut or Silhouette will make this process much faster and the final shapes more precise. For those with access to an Accucut die-cut device, Custom Shape Pros has the files and can make dies on request. Although this activity is designed for tiles made from paper, it could be modified for use with 3D-printed or laser-cut tiles.

3. Make patches containing eight tiles. Get seven hats with one orientation and one with the opposite orientation. Arrange them as shown in Figure 2, and tape them together with clear tape. Once one patch has been made, it may be quicker and easier to make the rest of the patches by placing the hats on top of the existing arrangement and using it as a template. Make 49 of these patches.

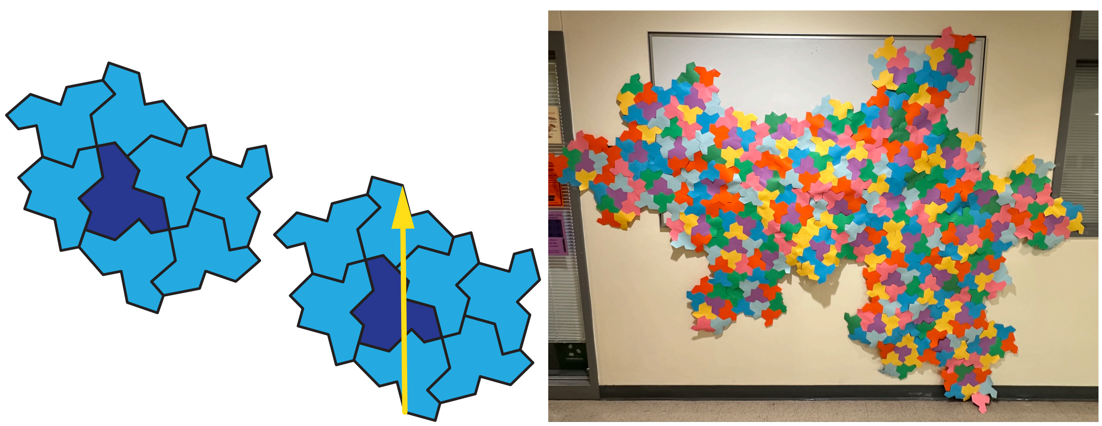

Figure 2: *Left: Patches of eight tiles with a reflected tile in the center. Add painter's tape with an arrow on it. Right: A tiling by hats made by students out of colored paper.*

4. Tape lines across the patches. Put painter's tape or masking tape on the 49 patches exactly—between these specific pairs of vertices—as in Figure 2. Draw an arrowhead at the end of the tape as shown.

[1]In fact, we will need only 329 in one orientation and 48 in the other orientation since we will remove some of the tiles during the construction.

5. Arrange these patches in groups of seven. Think of the painter's tape with an arrow at the end like the hand of a clock. Arrange seven patches as shown in Figure 3. In the bottom four patches, the painter's tape will take the shape of four sides of a hexagon—in left-to-right order, the arrows point to 12 o'clock, four o'clock, two o'clock, and 12 o'clock. The tape on the top three patches forms three sides of a hexagon with the arrows pointing, in right-to-left order, 12 o'clock, 10 o'clock, and eight o'clock. With the exception of one 12 o'clock patch, the tip of each arrow in the hexagonal path points to the tail of the next arrow.

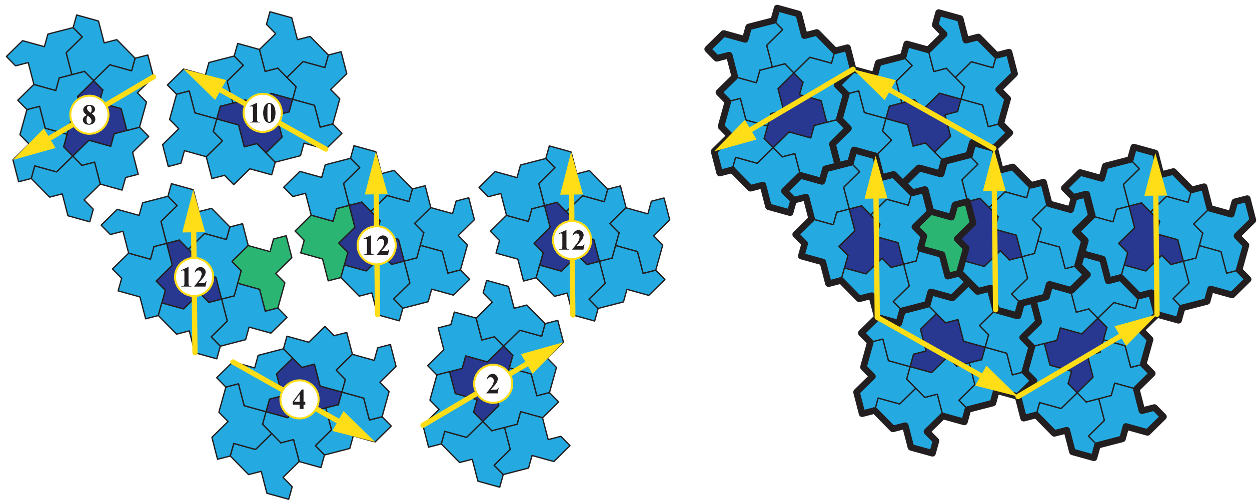

Figure 3: *The painter's tape on the patches forms parts of a regular hexagon. The tiles that are colored green will overlap each other.*

6. Tape the patches together. The patches should fit together perfectly with one exception—as shown in Figure 3, two tiles will overlap. One of these extra tiles can be removed. Tape the patches together. Then, make six more copies of this larger patch. Remove the painter's tape if it is possible to do so.

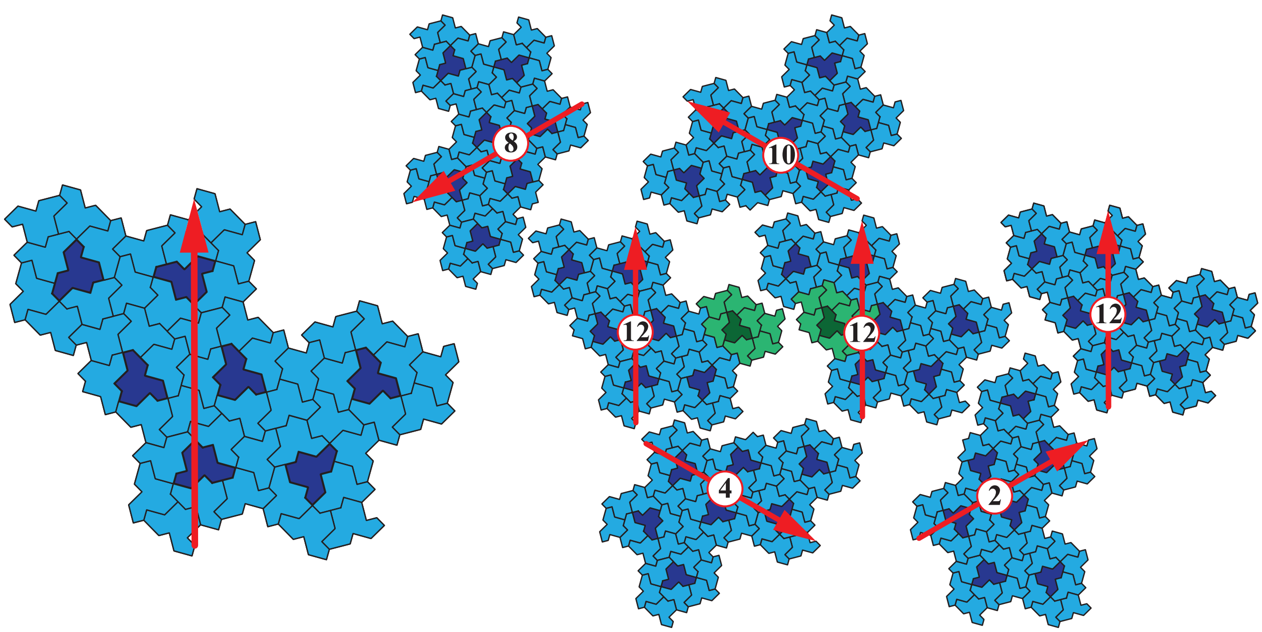

Figure 4: *Add painter's tape with an arrow on it to each patch. Arrange the seven patches as shown. The tiles that will overlap are colored green.*

7. Tape lines across the patches. Put painter's tape on these new larger patches as shown in Figure 4—between those specific pairs of vertices. If the old painter's tape is still on the smaller patches, it may be helpful to use tape of another color. Draw an arrow on the end of the tape as shown.

8. Arrange the seven patches. As in step 4, place the patches so the painter's tape forms the bottom of a hexagon in left-to-right order 12 o'clock, four o'clock, two o'clock, and 12 o'clock and the top of a hexagon, in right-to-left order, 12 o'clock, 10 o'clock, and eight o'clock. This arrangement is shown in Figure 4.

9. Tape the patches together. The patches should fit together with each piece of painter's tape lining up with the painter's tape on the neighboring patch. They should fit together perfectly except for one overlapping region. In Figures 4 and 5, we see that exactly two groups of eight tiles will overlap each other. One of these sets of eight tiles can be removed. Tape the patches together. If possible, remove the painter's tape.

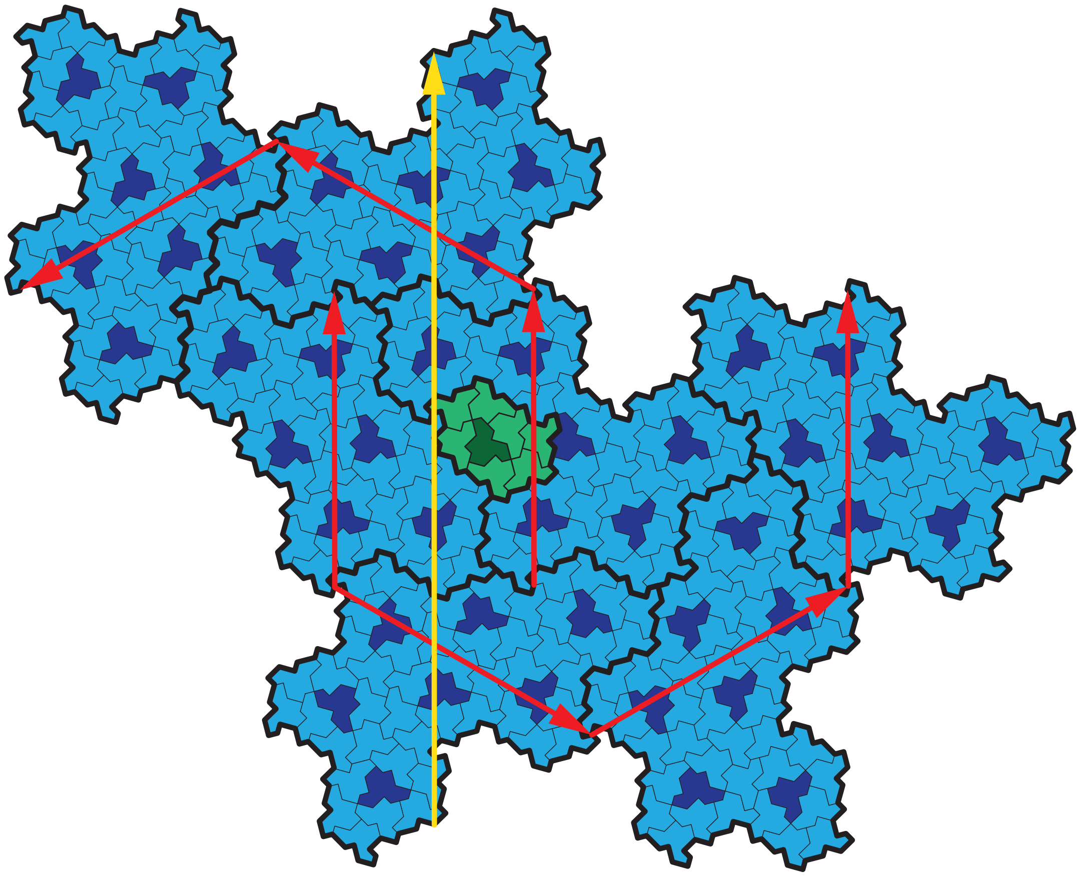

Figure 5: *The seven patches taped together with overlapping tiles (colored green) removed. The tape line for an even larger patch is indicated.*

References

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[3] D. Smith, J. S. Myers, C. S. Kaplan, and C. Goodman-Strauss. "An aperiodic monotile." 2023. https://arxiv.org/abs/2303.10798.