Enclave Depth in Hitomezashi Stitchery

D. Jacob Wildstrom

University of Louisville, Kentucky, USA; djwild01@louisville.edu

Abstract

The patterns of closed or open shapes produced by interleaving horizontal and vertical dashed lines, some of which are offset by a half period, are the fundamental designs behind the art of hitomezashi, the "one-stitch" style of the sashiko embroidery technique. These same patterns also appear in the AA Weaving technique developed by Ahmed. An "enclave" in a division of a finite canvas is a region which is wholly enclosed by other regions. The *order* or *depth* of an enclave is the number of regions one must pass through to reach the edge of the canvas. Some hitomezashi patterns possess very deep enclaves, but in general, the maximum depth of an enclave, even in fractal or random designs, is sharply limited.

Formalization of Hitomezashi

Hitomezashi is a form of embroidery done with a sequence of horizontal and vertical running stitches on a canvas. A running stitch progresses a certain length, which we might call a *unit*, on the top of the work, passes through the fabric, and then progresses a single unit on the bottom of the work, passes through the fabric, and then repeats. The appearance of a running stitch, from either side of the work, is of a dashed line, with equal amounts of dash and blank space. If several horizontal stitches are placed with a distance of a unit between them, and likewise several vertical stitches, the thread will form a grid, but if the stitches are running stitches as described here, exactly half of the grid will be visible from each side of the work. There are two different potential ways each stitch could be worked: starting on the top of the work, or starting on the bottom. The vertical and horizontal stitches can thus be fully characterized by a pair of bitstrings, in which we might conventionally regard a 1 as a stitch whose beginning is visible, and 0 as a stitch whose beginning is invisible, and read the instructions for horizontal stitches top-to-bottom, and vertical stitches left-to-right. These conventions are illustrated in Figure 1a, with the series of stitches corresponding to a horizontal-stitch bitstring 1100100 and a vertical-stitch bitstring 0100010100.

Several mathematical works have specifically considered the patterns achievable by hitomezashi, as well as the characteristics of traditional hitomezashi designs [2, 3, 5]. The regions described by hitomezashi are additionally realizable through a crafting process developed by Ahmed called AA Weaving [1]. Further mathematical works have considered the same pattern of gridlines as the result of a process described as "corner percolation", "site percolation", or "two-dimensional percolation" [4].

(a) *Hitomezashi described by the bitstrings 1100100 and 0100010100*

(b) *The "AA bitmap" resulting from 2-coloring regions*

Figure 1: *Hitomezashi designs and corresponding AA weaving patterns*

(a) *Yamagata, with enclave depth zero at any size* **(b)** *Pattern with a maximum enclave depth*

Figure 2: *Patterns of minimum and maximum enclave depth*

Enclaves in Hitomezashi

The AA weaving patterns described above differ from many of the other investigations by considering the way in which the gridlines subdivide the canvas into regions. The techniques of AA weaving produce not the gridlines of the embroidery, but the regions enclosed by the stitches, in alternating colors, as can be seen in Figure 1b. Notably, in this small example, there is only a single closed loop, visible, near the bottom center of the work. In larger works, however, it is common not only for sequences of incident stitches to form loops, but even for loops to be within loops. To explore this phenomenon, we shall consider the concept of *enclave order*.

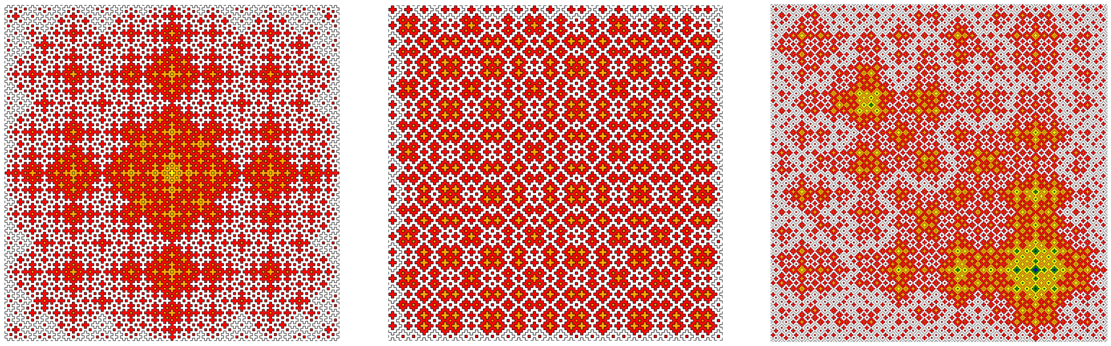
Definition 1. A region bounded by hitomezashi stitches which is not a closed loop is called an *order-zero enclave*; any region which is a closed loop is identified as a *order-*$(n+1)$ *enclave*, where n is the lowest enclave order of an adjacent region.

This terminology derives from political geography, in which a "second-order enclave" is a region belonging to one nation, surrounded by an entirely-enclosed enclave of another nation; an alternative definition for the order of an enclave is the number of stitches it is necessary to cross in order to reach the outside of the work. The *enclave depth* of a pattern is the maximum order of an enclave in the pattern.

By construction, it is easy to show that patterns of arbitrary size with zero enclave depth are possible, including the patterns Seaton and Hayes[5] identify as dan tsunagi, hirayama michi, and yamagata; an example of the yamagata form is exhibited in Figure 2a. It is also easy to construct patterns whose enclave depth is a quarter of the length of the shorter of the two bitstrings used to generate the pattern; Seaton and Hayes identify this as dual to a variant of the yamagata pattern, and an example is shown in Figure 2b. The enclave depth exhibited by this pattern is in fact the largest possible.

Proposition 1. *A hitomezashi pattern determined by length m and n bitmaps for $m \leq n$ has an enclave depth of at most $\frac{m+3}{4}$.*

Proof. We will start by demonstrating that if a hitomezashi pattern is coordinatized with the upper left corner at $(0,0)$, there is a path from the point $(x_0+\frac{1}{2}, y_0+\frac{1}{2})$ to an edge of the work which passes through at most $\frac{y_0}{2}+1$ stitches. If $x_0 = 0$, then we can obviously reach the left edge in a single step, crossing at most one stitch. Otherwise, let us consider two paths: a straight segment from $(x_0+\frac{1}{2}, y_0+\frac{1}{2})$ to $(x_0+\frac{1}{2}, -\frac{1}{2})$, and a path consisting of two segments from $(x_0+\frac{1}{2}, y_0+\frac{1}{2})$ to $(x_0-\frac{1}{2}, y_0+\frac{1}{2})$ and then on to $(x_0-\frac{1}{2}, -\frac{1}{2})$. In total, these two paths will cross at most y_0+2 segments: exactly one of these two paths will cross a horizontal stitch at each of horizontal lines $y=0, y=1, \ldots, y=y_0$, since for each y, exactly one of the two segments (x_0-1, y) to (x_0, y) or (x_0, y) to (x_0+1, y) is present, and it is also possible that the second path crosses the vertical stitch from (x_0, y_0) to (x_0, y_0+1). Since these two paths collectively cross y_0+2 stitches, one of the paths crosses no more than $\frac{y_0+2}{2}$ stitches.



(a) *Fibonacci Snowflake generated by the Pell word $u_6\widetilde{u_6}$* **(b)** *Pattern generated by 128 bits of the Thuy-Morse sequence* **(c)** *Pattern generated by 234 terms of the Kolakoski sequence*

Figure 3: *Self-similar hitomezashi designs, with color-coded enclave depth*

Returning to the question of enclave depth on a whole pattern, by applying this same theorem to four rotations of a finite canvas, it is clearly possible to reach the outer edge of a $(m-1)\times(n-1)$ canvas from a point $(x_0+\frac{1}{2}, y_0+\frac{1}{2})$ via a path crossing no more than the minimum of $\frac{x_0+2}{2}$, $\frac{y_0+2}{2}$, $\frac{m-1-x_0-2}{2}$, $\frac{n-1-x_0-2}{2}$ stitches. This quantity is maximized by when $x_0 = \lfloor\frac{m-1}{2}\rfloor$ and $y_0 = \lfloor\frac{n-1}{2}\rfloor$, which gives a maximum upper bound on the enclave order of $\frac{m+3}{4}$. □

Enclave Depth in Self-similar and Random Patterns

Many of the traditional patterns Seaton and Hayes identify have constant enclave depth regardless of work size; the yamagata variant shown in Figure 2b is an exception in that regard. However, nontraditional patterns exhibiting self-similarity can be produced by generating the defining bitstrings for a hitomezashi design using a recursive procedure. Seaton and Hayes define one such pattern based on a recursively-defined binary sequence called a Pell word, with the generative rule:

$$u_0 \text{ is the empty string}, u_1 = 1, u_n = \overline{u_{n-1}\overline{\widetilde{u_{n-2}}}}u_{n-1} \text{ for } n \geq 2$$

where $\overline{x}$ is the binary complement of the bitstring x, and $\widetilde{x}$ is the reversal of x. Using $u_n\widetilde{u_n}$ as both binary codes for a hitomezashi pattern creates "Fibonacci snowflakes" whose complexity increases with n. Enclave depth in the Fibonacci snowflake appears to increase so that the order-n snowflake has enclave depth $\frac{n+1}{2}$. The order-6 snowflake has enclave depth 3, which can be visualized in Figure 3a.

There are other notable recursively defined bitsequences which can produce other self-similar hitomezashi patterns, including the Thuy-Morse sequence and the Kolakoski sequence. Hitomazashi patterns for these two can be seen in Figures 3b and 3c. The Thuy-Morse sequence never appears to produce a pattern with an enclave depth greater than 2, while the Kolakoski sequence exhibits a roughly logarithmic enclave depth: 3 terms are enough to induce depth 1, 11 for depth 2, 44 for depth 3, 89 for depth 4, and 234 for depth 5.

In addition to self-similar patterns, one way to step outside of traditional design sensibilities is with randomness. The most natural way to do so is by selecting the defining bitstrings for a design entirely at random. The enclave depth of a design given by two random bitstrings of lengths $m \leq n$ will be a random variable, with nonzero probabilities of each value from 0 to $\lfloor\frac{m+3}{4}\rfloor$. For very small values of m and n, the complete probability distribution can be determined; for larger values, since there are 2^{m+n} different bitstring pairs possible, exhaustively testing all bitstrings becomes prohibitively computationally intensive.

The enclave depths resulting from a large number of tests of random bitstring pairs of various sizes appear in Table 1. A clear conclusion to reach from this table is that the larger dimension of a random pattern has almost no effect on the distribution of enclave depth: 20×20 patterns have very nearly the same depth

Table 1: *Enclave-depth probability distribution of random bitstrings*

m	n	# trials	Mean	0	1	2	3	4	5	6	7
5	5	2^{10}	0.633	36.7%	63.3%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
5	10	2^{15}	0.851	14.9%	88.9%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
10	10	2^{20}	1.003	1.9%	95.8%	2.3%	<0.1%	0.0%	0.0%	0.0%	0.0%
20	20	100000	1.331	<0.1%	67.4%	32.1%	0.5%	0.0%	0.0%	0.0%	0.0%
20	40	100000	1.332	0.0%	67.4%	32.2%	0.5%	<0.1%	0.0%	0.0%	0.0%
20	100	100000	1.330	<0.1%	67.5%	32.0%	0.5%	0.0%	0.0%	0.0%	0.0%
20	200	100000	1.330	<0.1%	67.5%	31.9%	0.5%	<0.1%	0.0%	0.0%	0.0%
40	40	100000	2.076	0.0%	9.4%	74.6%	15.3%	0.8%	<0.1%	0.0%	0.0%
100	100	100000	3.380	0.0%	<0.1%	10.2%	51.2%	30.4%	7.0%	1.1%	0.1%

distribution as 20×200 patterns. It also appears to be the case that enclave depth tends to increase with smaller-dimension size, a result which can be quantified.

Theorem 1. *For positive integers $m \leq n$ and k, the probability that a hitomezashi pattern generated by length m and n uniformly random bitstrings has enclave depth of k or more is at least $\left(1 - (1 - 2^{-2k})^{\lfloor m/k \rfloor}\right)^2$.*

Proof. Generalizing the dual-yamagata pattern in Figure 2b, it is clear that if the sequence $(01)^{k-1}00(10)^{k-1}$ appears starting at an odd index in both bitstrings (with the first index being 0), then the resulting hitomezashi pattern has an enclave of order k. The probability of that particular pattern occurring at a specific index of a bitstring is 2^{-2k}; the probability that it occurs at position $1, 1+2k, 1+4k, \ldots,$ or $1+2(x-1)k$ of a bitstring of length at least xk is $1-(1-2^{-2k})^x$, since each of these events is independent. Thus, there is a probability of at least $1-(1-2^{-2k})^{\lfloor m/k \rfloor}$ that a random bitstring of length m contains this sequence starting at an odd index, and likewise probability $1-(1-2^{-2k})^{\lfloor n/k \rfloor}$ for a bitstring of length n, and so there is a probability of at least $\left(1-(1-2^{-2k})^{\lfloor m/k \rfloor}\right)\left(1-(1-2^{-2k})^{\lfloor n/k \rfloor}\right) \leq \left(1-(1-2^{-2k})^{\lfloor m/k \rfloor}\right)^2$ that both bitstrings contain this sequence starting at an odd index, which ensures an enclave of order k. □

This bound is quite sloppy, using individual independent blocks to make the calculation easier when the actual probability of the target subsequence is surely much higher. Nonetheless, this result suffices to show that, for any specific k, enclave depth exceeding k almost always occurs as m and n increase without bound.

References

[1] A. G. Ahmed. "AA Weaving." *Proceedings of Bridges 2013: Mathematics, Music, Art, Architecture, Culture*. G. W. Hart and R. Sarhangi, Eds. Phoenix, Arizona: Tessellations Publishing, 2013. pp. 263–270. http://archive.bridgesmathart.org/2013/bridges2013-263.html.

[2] C. Defant and N. Kravitz. "Loops and regions in Hitomezashi patterns." *Discrete Mathematics*, vol. 347, no. 1, 2024, p. 113693. https://www.sciencedirect.com/science/article/pii/S0012365X23003795.

[3] C. Hayes and K. Seaton. "A Two-Dimensional Introduction to Sashiko." *Proceedings of Bridges 2020: Mathematics, Art, Music, Architecture, Education, Culture*. C. Yackel, R. Bosch, E. Torrence, and K. Fenyvesi, Eds. Phoenix, Arizona: Tessellations Publishing, 2020. pp. 517–524. http://archive.bridgesmathart.org/2020/bridges2020-517.html.

[4] G. Pete. "Corner percolation on $\mathbb{Z}^2$ and the square root of 17." *The Annals of Probability*, vol. 36, no. 5, 2008, pp. 1711 – 1747. https://doi.org/10.1214/07-AOP373.

[5] K. A. Seaton and C. Hayes. "Mathematical specification of hitomezashi designs." *Journal of Mathematics and the Arts*, vol. 17, no. 1-2, 2023, pp. 156–177.