Kagome Gyroid

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Abstract

A model of the gyroid surface can be made using a basketry technique. It is based on a simple triangulation that has faces close enough to equilateral to form the basis of a kagome structure. In principle this method is quite straightforward but there are several practical difficulties, which can be resolved using a range of strategies. The resulting structure relates to some other models of the gyroid.

Introduction

Over ten years ago I had considered arrangements of helices, including the possibility of having structures with both right- and left-handed ones [2], but I did not recognize how they might relate to the gyroid surface. Duston Wetzel's statement in the Bridges 2023 art exhibition [8] first drew my attention to how this might be possible.

I should have realised the connection when I found another derivation of the "discrete gyroid" that Reitebuch, Skrodzki and Polthier described in their Bridges 2019 paper [6]. Figure 1 shows what I missed. My paper [3] relates arrangements of helices to regular cylinder packings [5], and one of them (Figure 7 there), which is the same structure as Figure 9 in [2], is reproduced here without cylinders (Figure 1(a)). Figure 1(b) shows the curvilinear hexagon that is triangulated in the discrete gyroid. Figure 1(c) shows how it can tessellate with 4-valent vertices. Alan Schoen has described this construction in greater detail [7].

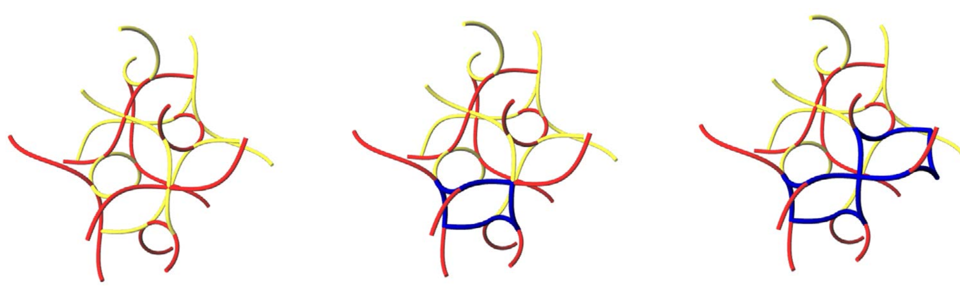

Figure 1: *(a) Intersecting helices around the cylinders of the *Π cylinder packing. (b) A curvilinear hexagon includes arcs from helices of both senses. (c) The hexagons tessellate to cover the gyroid.*

Duston was thinking in terms of geodesics on the gyroid. Following several conversations I started to think about what other geodesics might be interesting. One idea was that they could be strands in a basket structure.

Construction

I had actually tried to make a model of the surface using plastic strapping when I first came across the "discrete gyroid" triangulation in 2017 [1]. I gave up very quickly because the material I use is quite slippy and has significant stiffness, so the structure is quite unstable, especially during the early stages of construction, and the clips I was using were not strong enough. Felicity Wood recently drew my attention to "Wonder Clips", used by quilters, which are certainly strong enough but they are rather too big. Embroiderers use a smaller version called "Mini Wonder Clips" that work perfectly. I do not think I could have made much progress without them.

The basic structure in kagome is a hexagon (Figure 2(a)), which is why it is often called hexagonal open weave or hexweave. If only these elements were present a flat woven sheet would result. In traditional basketry, “corners” are introduced at particular points by reducing the number of strands, usually to five, to create an approximately conical region. In creating a model of the gyroid all the “corners” are produced by adding two more strands to the hexagon to form saddle-shaped octagons (Figure 2(b)). Left to itself such a corner would relax into a flat octagon so it must be held in place tightly with clips. Conversely the hexagons are flat when tightly woven so they must be quite loose in order to achieve the required monkey-saddle shape in the model. This inconsistency does not seem to cause any problems.

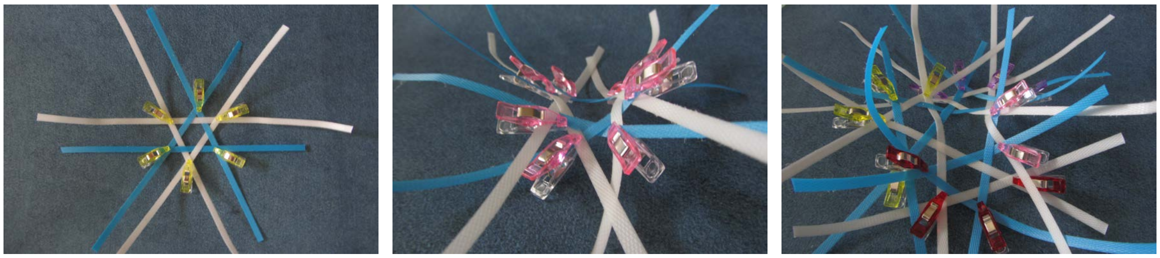

Figure 2: *(a) The standard hexagon. (b) An octagonal “corner”. (c) An octagonal corner introduced between hexagons.*

Locally there are never three hexagons around a single triangle, and octagonal corners are constructed wherever two hexagons are adjacent in the basket structure. Figure 2(c) shows a corner between a hexagon (held by yellow clips) and another (held by red clips). There is another hexagon with purple clips that are hardly visible. It would be quite straightforward to continue like this, resulting in a model of the hyperbolic plane similar to those created using crochet. To produce a gyroid the structure needs to close in on itself forming tunnels. In principle this seems easy but it is surprisingly difficult to decide exactly which of the numerous loose ends should be woven together to close things up, rather than just adding more strands.

Keeping Track

There are some signposts that can be introduced to make it easier to find a way through the apparent chaos. The simplest is to use two colours of strand as in Figure 2. Even before I started building anything I conjectured that any particular strand would follow a chiral path. The gyroid is not chiral, so there must be an equal number of strands of each handedness, and they can be coloured accordingly. Symmetry requires every hexagon to have three strands of each handedness, and they must are arranged alternately. Having two colours makes it immediately obvious whether to add a hexagon or an octagon at any position. Figure 2 shows how: at a hexagon strands of the same colour cross (yellow and red clips); at an octagon both colours are present at a crossing (pink clips).

The discrete gyroid triangulation (Figure 3) provides some more help. There are vertices of two types: 6-valent, corresponding to hexagons in the model, and 8-valent, corresponding to octagons. The 6-valent vertices line up along axes of 3-fold symmetry, which lie in four different directions (parallel to the space diagonals of a cube). Figure 3(b) shows the corresponding triangulated skew hexagons coloured according to these directions. Fortunately the clips come in different colours allowing hexagons in the model to be similarly identified according to their orientation. Two observations are particularly helpful: each colour is present where four hexagons meet at an 8-valent vertex (or around an octagon in the model); there are only two colours that alternate in going in any straight line from hexagon to hexagon through parallel edges in the triangulation, which correspond with opposite crossings in the model. These rules must be followed when existing loose ends are woven to form a tunnel.

It is helpful to introduce dowels through the hexagons along the 3-fold axes (Figure 3(c)). This helps to keep things in shape as well as helping to identify the colour of hexagons, even without clips. The figure actually shows an early stage during the construction. The dowels were quite heavy and kept falling out. I later replaced them with balsa and stuck coloured dots on the ends, which also helped to make identification easier.

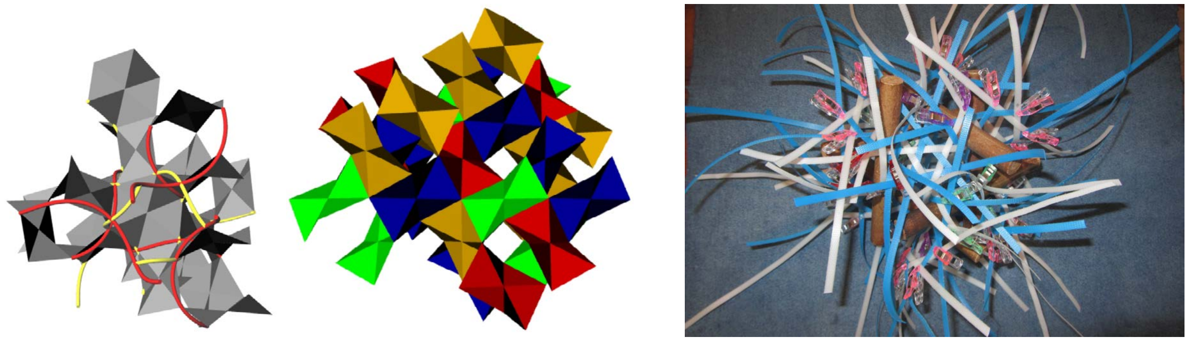

Figure 3: *(a) The "discrete gyroid" with the helices of figure 1. (b) The hexagons coloured according to orientation. (c) A stage during construction looking down an axis of threefold symmetry.*

The gyroid is an infinite surface but any model must come to an end at some stage. The bigger it is the more loose ends there will be, and they need to be anchored in some way. I found that passing a hot needle through overlapping strands created enough of a weld to keep things in place.

Discussion

This is the first basketry construction that I have attempted where the Gaussian curvature is negative during every stage. Even models of the Schwarz P and D surfaces can be made using convex units analogous to the truncated octahedra and truncated tetrahedra of the corresponding infinite polyhedra. The small loops (equators) in each unit cause problems, and fitting them together is not without technical difficulties, in one case probably insurmountable [4], but there is never any need to control saddle corners. In this case there were some new challenges and overcoming them was one of my motivations in attempting it. Another was the possibility of gaining fresh insights into aspects of the gyroid.

Figure 4(a) shows the completed model looking down tunnels that correspond with the *Π cylinder packing [5]. The tunnels seen in Figure 4(b) correspond with the *Σ cylinder packing.

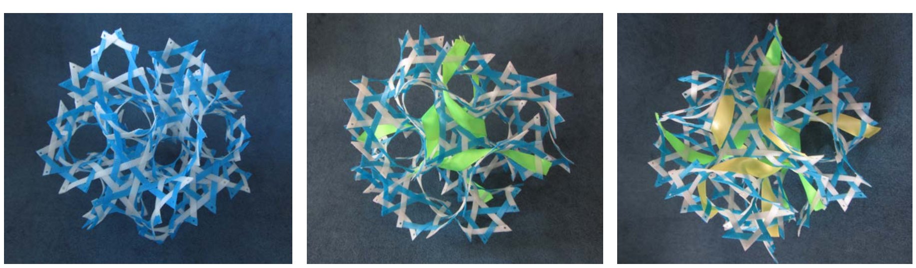

Figure 4: *(a) The completed model. (b) Ribbons sitting in the saddle octagons that pass over white strands. (c) Ribbons added that pass over blue strands in the saddle octagons.*

Figure 2(b) shows a view of a saddle octagon looking along a line through two crossings of blue strands. In the completed model ribbons can sit in these saddles, and they follow helical paths that pass over crossings of strands that are always the same colour (Figure 4(b,c)). The green ribbons in Figure 4(b) lie on one side of the surface; the yellow ribbons that have been added in Figure 4(c) lie on the other side. They wind around the tunnels visible in Figure 4(a) and correspond with the helices in Figure 1. Although in the figures they appear to be parallel to the edges of hexagons the strands of the basket must deviate from those helical paths. The curvilinear hexagons in Figure 1 tessellate by themselves to cover the surface. There could be no octagons if the strands did not deviate from the helices.

The kagome model is closely related to another infinite polyhedron that is an approximate gyroid. It consists of regular hexagons and 80.406° rhombi. The saddle octagons correspond to groups of four rhombi and the relation between the hexagons in both models is obvious. Figure 5(a) shows this polyhedron coloured to show the grouping. Six helices like those in Figure 1 are shown in Figure 5(b). They pass through vertices common to the four rhombi at the centre of the groups, defining the edges of a curvilinear hexagon. Figure 5(c) shows how this model relates to the "discrete gyroid" of figure 3.

Figure 5: *(a) Another polyhedral gyroid. (b) Six helices from Figure 1 defining a curvilinear hexagon. (c) Some of the "discrete gyroid" of Figure 3 added for comparison.*

This polyhedron could not be the basis of a different basketry model since there are neither faces nor vertices that could give rise to saddle-shaped elements, which are needed to produce negative Gaussian curvature. Curvilinear hexagons (monkey-saddles) can only occur if the naturally flat hexagons in a woven structure are distorted by some other elements, so the tessellation indicated in Figure 1(c) could not be used either. The only variation that seems possible would be to add more hexagons, equivalent to further dividing the triangles of the discrete gyroid in Figure 3, which would create a more relaxed structure.

References

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