Playing with Connections and Variations: Golden Sierpinski Spirals

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Abstract

This paper explores playing with different mathematical concepts in the creation of visually appealing objects. The concepts involved are the golden ratio, the Sierpinski gasket, and spirals. The golden ratio and spirals have both been a source of inspiration for mathematicians and artists for centuries. The Sierpinski gasket and variations are used. The goal is to explore different ways to create spirals made from variations of the Sierpinski gasket and that are also connected to the golden ratio.

Introduction

What does it mean to be a "golden Sierpinski spiral"? This paper is all about playing with connections and variations to figure out what objects could be considered as 'golden Sierpinski spirals', see Figure 1.

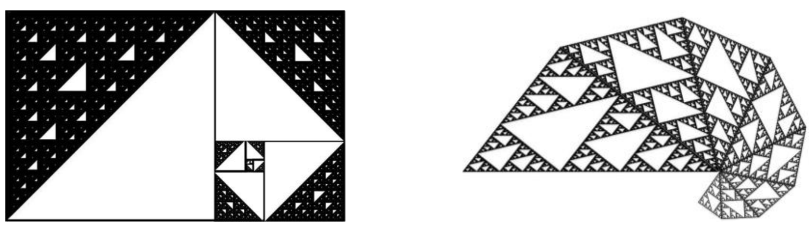


Figure 1: *Two different 'golden Sierpinski spirals'.*

At Bridges conferences, I am often struck by how many presenters mention "playing". Francis Su's wonderful book *Mathematics for Human Flourishing* includes a chapter on play [16]: "That's what doing math looks like when you learn any new idea- you play with it. Even for professional mathematicians, the beginning of a research project is playful exploration: contemplating patterns, playing with ideas, exploring what's true, and enjoying the surprises that come along the way." Indeed, play is relevant for our survival as a species, as in a recent article in National Geographic [7]: "the urge to play underlies most of humanity's greatest inventions, artworks, and scientific breakthroughs".

The golden ratio, here denoted by φ, is well-known because it arises in many areas of math and art [4][11][14][18]. φ is the positive root of the equation $x^2 - x - 1 = 0$, thus $\varphi = (1 + \sqrt{5})/2 \approx 1.618$ and $\varphi^2 = \varphi + 1$. A rectangle that can be split into a square and a smaller rectangle with the same proportions as the original rectangle is a *golden rectangle*.

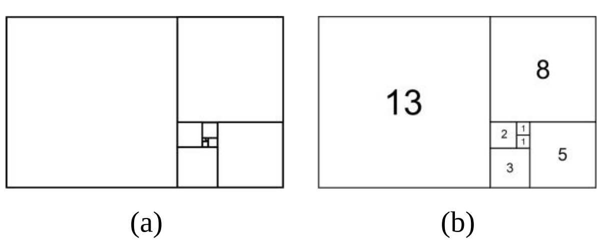
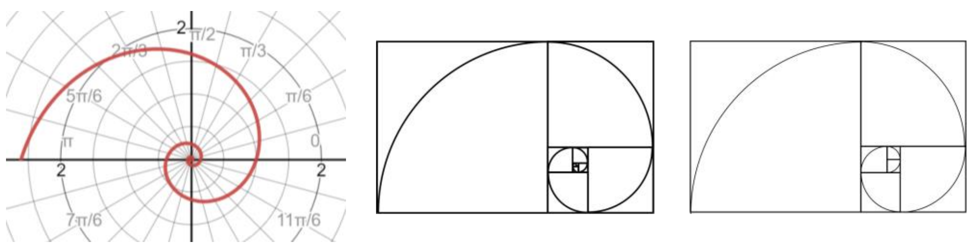


Figure 2: *(a) Golden rectangle split into successive squares forming smaller golden rectangles; (b) golden rectangle approximation using squares with side lengths corresponding to Fibonacci numbers.*

Any rectangle whose ratio of length to width is φ is a golden rectangle. A square is a 'gnomon' for a golden rectangle. A *gnomon* of an object is an object which when added to the original object results in a new object that is similar to the original object [8]. The Fibonacci numbers F_n, where $F_1 = 1, F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, are connected to the golden ratio. Ratios of successive Fibonacci numbers are: $1/1 = 1,\ 2/1 = 2,\ 3/2 = 1.5,\ 5/3 = 1.\overline{6}$ and the limit as $n \to \infty$ of F_{n+1}/F_n is φ. Figure 2(a) displays a golden rectangle split into successive squares and Figure 2(b) display a golden rectangle approximation using Fibonacci numbers.

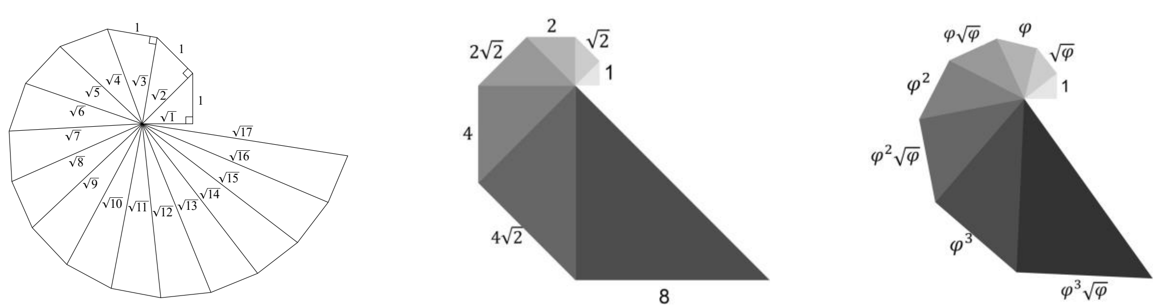
Spirals are found throughout nature, art, and math [3][5][10]. There are many different mathematical models [5][6]. This paper focuses on logarithmic spirals and the Spiral of Theodorus. The word spiral can be used as a noun, a verb, an adjective, or an adverb. A beautifully illustrated reference is *The Curves of Life*: "a logarithmic spiral…is as near as we can get in mathematics to an accurate definition of the living thing. Nor does the mathematician fare any better when he tries to express beauty in terms of measurement. In other words, the baffling factor in a natural object is its life; just as the baffling factor in a masterpiece of creative art is its beauty. May it not then be true that beauty, like life and growth, depends not on exact measurement or merely mathematical reproduction, but on those subtle variations…" [3].



(a) (b) (c)

Figure 3: *(a) Golden logarithmic spiral; (b) Golden spiral; (c) Fibonacci spiral.*

A logarithmic spiral can be modelled using the polar equation $r = ae^{k\theta}$. It is a self-similar curve that is often used to describe natural objects (seashells, galaxies, hurricanes, etc.) [3][9]. The distance between turnings grows in geometric progression. Figure 3(a) displays the logarithmic spiral $r = \varphi^{2\theta/\pi}$. The distance from the origin grows by a factor of φ for each quarter turn. One way to approximate this golden logarithmic spiral is with the *golden spiral*. This curve is formed in a golden rectangle that has been split into squares by joining circular arcs that go from one corner of a square to the opposite corner as in Figure 3(b). Another way to approximate the golden logarithmic spiral is with a *Fibonacci spiral*. Now the circular arcs go through squares with side lengths that correspond to Fibonacci numbers, see Figure 3(c).



(a) (b) (c)

Figure 4: *(a) Spiral of Theodorus [15]; (b) variation with right isosceles triangles; (c) variation with Kepler triangles.*

The Spiral of Theodorus consists of right triangles beginning with a triangle with sides of length 1, see Figure 4(a) [5][6][10]. Each successive triangle is a right triangle with one side of length 1 and the other side has length given by the hypotenuse of the previous triangle. Variations of this spiral use different right triangles, where the side length of one side is equal to the hypotenuse of the previous triangle. One variation is with isosceles right triangles so the lengths increase with factor of $\sqrt{2}$, see Figure 4(b). Figure 4(c) displays a variation with Kepler triangles. A *Kepler triangle* is a right triangle whose sides are in geometric progression [17]. Denote the length of the shortest side by k and the ratio of the progression by $\sqrt{x}$, thus $(k)^2 + (k\sqrt{x})^2 = (kx)^2$. Recall that $1 + \varphi = \varphi^2$. This forces the ratio to be $\sqrt{\varphi}$.

The Sierpinski gasket is a well-known fractal whose boundary is a triangle (either equilateral or right isosceles) [1][12]. This paper uses the right triangle version because they are conducive to making spirals. The gasket can be generated from an iterated function system (IFS) [1]. An *iterated function system* (IFS) is a collection $\{f_1, f_2, \ldots, f_m\}$ where each f_i is a contractive mapping from the plane to the plane. A given IFS has a unique attractor A that satisfies $A = f_1(A) \cup f_2(A) \ldots f_m(A)$ [1]. Thus A is made of smaller versions of itself. Starting with any compact set X, form a sequence of approximations $\{A_n\}$, for $n \geq 0$, as follows. $A_0 = X$ and for $n \geq 1$:

$$A_n = \cup_{i=1}^{m} f_i(A_{n-1}) = f_1(A_{n-1}) \cup f_2(A_{n-1}) \cup \ldots \cup f_m(A_{n-1}).$$

The limit of the approximations as $n \to \infty$ is A. X is typically chosen to encompass the attractor, often by including the convex hull of the fixed points of the IFS mappings. The right triangle Sierpinski gasket is the unique attractor of the IFS $\{f1, f2, f3\}$ as follows [1]. Let T be the isosceles right triangle with vertices (0,0), (1,0), and (0,1). All three maps apply a contraction by a factor of $1/2$. The map f_2 shifts the contracted triangle to the right by ½ and the map f_3 shifts the contracted triangle up by ½. Figure 5 displays T, the first three iterations after applying the IFS to T, and the Sierpinski gasket as the unique attractor.

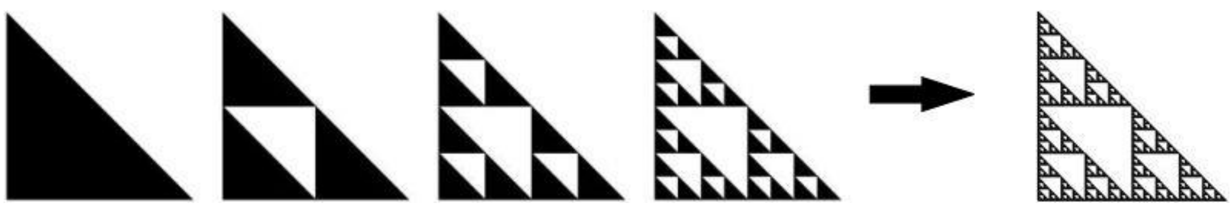


Figure 5: *IFS with Sierpinski gasket as unique attractor.*

Golden Sierpinski Spirals

There is more than one way to answer the question "What is a golden Sierpinski spiral?". This is part of the beauty of playing with connections and variations. The creations here are spiral in the sense that they start with an object and join it with scaled versions of the object that spiral around in some way. The starting object is a gnomon for the spiral. The first attempt is to spiral the Sierpinski gasket around squares with lengths given by Fibonacci numbers. There are two versions, an outer and inner spiral depending on where the Sierpinski gaskets are placed, see Figure 6. This attempt was not satisfying to me because the objects don't line up in a way that objects associated with the golden ratio typically do. Moreover, the copies of the gaskets don't have a common scaling ratio because they scale with ratios of Fibonacci numbers.

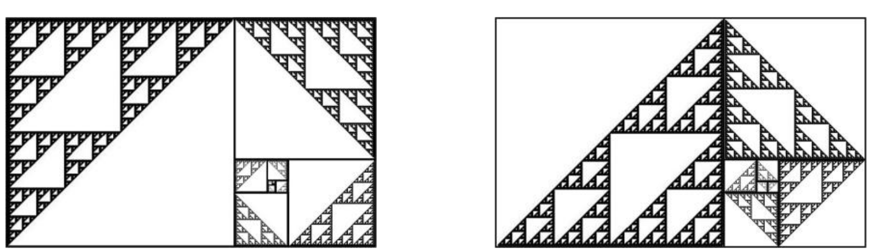
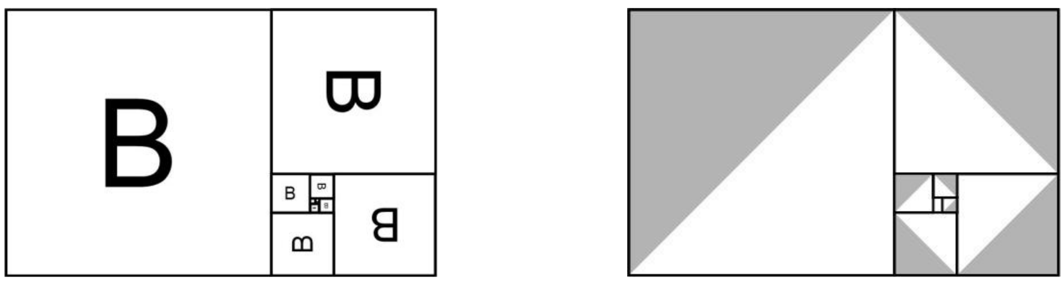


Figure 6: *Fibonacci spirals with Sierpinski gaskets.*

Consider the golden rectangle with length φ, width 1, and lower left corner at the origin (0,0). One can describe spirals in this golden rectangle using similarities. Define the *golden spiral map* s to be the map that consists of contraction by a factor of $1/\varphi$, rotation by 90° clockwise, and translation by φ in the horizontal direction and 1 in the vertical direction. Let B be any subset of the unit square. We can obtain a golden spiral associated with B by applying the spiral map s to B ad infinitum as in Figure 7(a); denote this by $S(B)$. $S(B)$ is not self-similar because it cannot be expressed in terms of only scaled down versions of itself but it can be expressed as the union of a scaled down version of itself with the original starting object.

$$S(B) = B \cup s(B) \cup s^2(B) \ldots = B \cup s\big(S(B)\big).$$

Recall that T is the triangle with vertices (0,0), (1,0) and (0,1). Let s_O be the map that consists of rotation by 90° clockwise followed by a vertical shift of 1 unit up. Let s_I be the map that consists of rotation by 90° counter-clockwise followed by a horizontal shift of 1 unit to the right. Figure 7(b) displays $S(s_O(T))$ in grey and $S(s_I(T))$ in white. The subscripts are "O" for outer and "I" for inner. Now instead of the triangle T we can start with the Sierpinski gasket. Figure 8 displays the inner and outer versions.



(a) $S(B)$ (b)

Figure 7: *(a) golden spiral associated with subset B; (b) triangles spiraling though a golden rectangle*

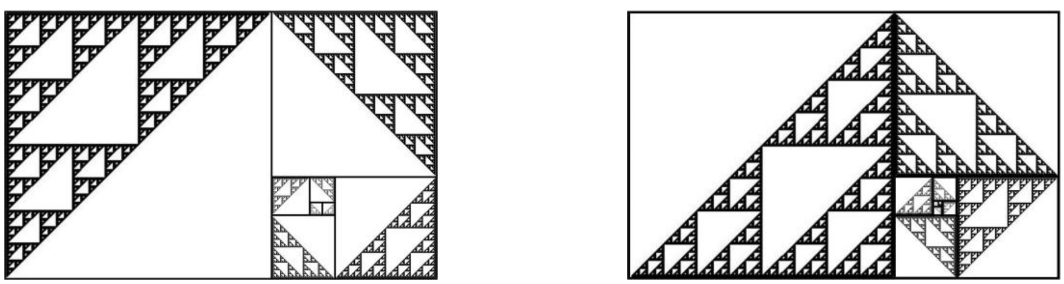


Figure 8: *Golden spirals with Sierpinski gaskets.*

The Sierpinski gasket does not scale according to the golden ratio, so consider a variation of the gasket that does scale with φ. To describe this variation called the 'golden gasket', use an IFS $\{g1, g, g3\}$ [2]. Start with the same triangle T but now the contraction factors are all $1/\varphi$. The map g_2 shifts to the right by $1/\varphi^2$ while the map g_3 shifts up by the same amount. Figure 9 displays the approximation A_1 to show the overlap, approximations A_2 and A_3, and the golden gasket.

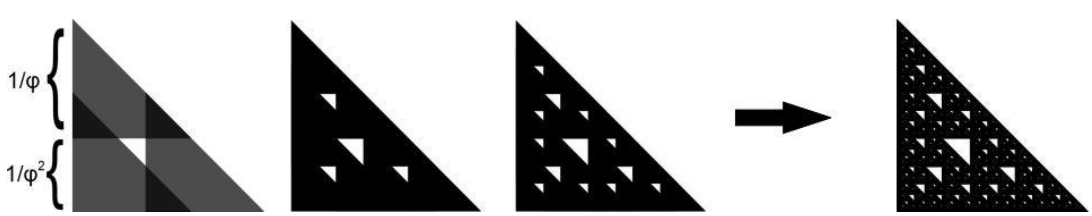


Figure 9: *Approximations leading to the golden gasket.*

Figure 10 displays the outer and inner spirals associated with the golden gasket. Now the holes of the golden gaskets line up in a much more visually pleasing way.

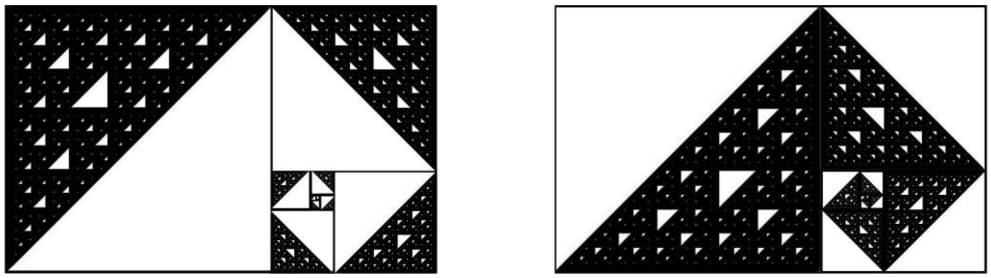


Figure 10: *Golden spirals with golden gaskets.*

Another way to spiral triangles around is following the method of the Spiral of Theodorus and its variations (see Figure 4). These spirals are built with right triangles. To build 'golden Sierpinski spirals' in this manner, we need right triangles that are connected to the golden ratio. We have seen that the golden gasket is a variation of the Sierpinski gasket that involves the golden ratio. There is another 'golden' variation of the Sierpinski gasket that involves a different right triangle. Recall that a Kepler triangle is a right triangle whose sides are in geometric progression [17]. Recall that $1 + \varphi = \varphi^2$. Let K denote the Kepler triangle with vertices (0,0), (1,0) and $(0, \sqrt{\varphi})$, see first image of Figure 11. The triangle K can be mapped to 3 smaller Kepler triangles that are subsets of K as follows. The first map k_1 has a horizontal contraction factor of $1/\varphi$ while the vertical contraction factor is $1/\varphi^2$. The map k_2 has a contraction factor of $1/\varphi^2$ and a horizontal shift of $1/\varphi$. The map k_3 has a contraction factor of $1/\varphi$ and a vertical shift of $1/\varphi\sqrt{\varphi}$. The second image in Figure 11 shows the 3 maps applied to K: $k_1(K)$ is the lower left triangle, $k_2(K)$ is the lower triangle on the right, and $k_3(K)$ is the upper triangle. The IFS $\{k_1, k_2, k_3\}$ has a unique attractor, call it the *Kepler gasket*.

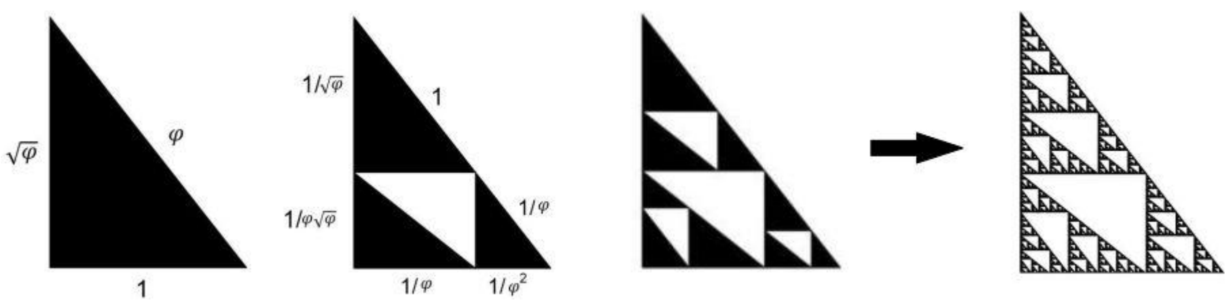
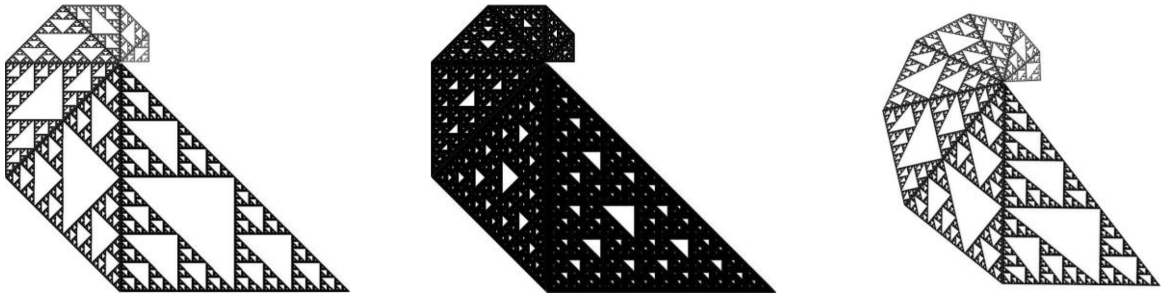


Figure 11: *Triangle K and first two approximations leading to Kepler gasket.*

Figure 12 displays three variations on the Spiral of Theodorus. Figure 12(a) displays one variation that is made with Sierpinski gaskets, so is not connected with the golden ratio. This is included for comparison and to show that one can create visually appealing objects without the golden ratio. Figure 12(b) shows a variation that is made with golden gaskets. The boundary triangles of the Sierpinski gaskets and the golden gaskets are right isosceles triangles, thus there is a common geometric progression for all sides with a factor of $\sqrt{2}$. Figure 12(c) shows a variation made with Kepler gaskets. In this case, the boundary triangles of the Kepler gaskets are Kepler triangles. Here the shorter sides stay on the outside of the spiral so there is a common geometric progression for all sides with a factor of $\sqrt{\varphi}$. These spirals are not technically self-similar, they cannot be expressed as a union of only smaller versions of themselves. However, as with the spirals in Figures 7, 8, and 10, we can express these as a union of the starting gasket and a scaled down spiral. As mentioned above, the starting gasket acts as a gnomon for the spiral.



(a) (b) (c)

Figure 12: *Variations of the Spiral of Theodorus: (a) with Sierpinski gaskets; (b) with golden gaskets; (c) with Kepler gaskets.*

Summary and Conclusions

The golden ratio, Sierpinski gasket, and spirals are interesting on their own. This paper has presented ways to play with connections and variations to produce some examples of what can be called golden Sierpinski spirals. One could use a "better golden rectangle" instead [16]. Oher ways could include the golden triangle (used for its corresponding logarithmic spiral or to create a variation of the Sierpinski gasket). Another approach could use a different mathematical model for a spiral. Playing with connections and variations helps us to gain a deeper understanding of the concepts and to appreciate their beauty.

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