Visual Representations of Positive Integers using Geometric Patterns

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Abstract

This paper describes a series of artworks representing positive integers from 1 to 200 using geometric patterns of simple visual motifs. This is part of an ongoing series of artworks that use geometric patterns to give visual representations of the positive integers. The patterns used were chosen to visually highlight some properties of each number. The designs use a variety of motifs and arrangements to provide a diverse yet cohesive collection. One application of these patterns is as a teaching tool for helping students recognize and generalize patterns and sequences.

Introduction

Natural numbers play a fundamental role in many areas of mathematics and in our lives. We are taught to count at a very early age through native languages and base 10 representations. We learn basic number concepts such as even and odd numbers. In addition to counting, we also use numbers for labeling items and for making measurements. These usages give us a practical model of the concept of number. However, the abstract nature of numbers has prompted the investigation of how we can rigorously define them. For simplicity, I will refer to the positive integers as numbers in this paper.

Numbers have also been the subject of artists. From 1965–2011 Roman Opałka created his work OPALKA 1965/1–∞, consisting of 233 canvases with the numbers from 1 to 5,607,249 [2]. Robert Indiana used numbers as the subject or titles of many of his works, stating “Numbers fill my life. They fill my life even more than love.” [1]. Margaret Kepner, Jordan Booth, and Dan Bach, have exhibited work at Bridges or the Joint Mathematics Meetings have also had number as a themes in their works. Other artists have explored visual enumerations of mathematical concepts. At Bridges or the Joint Mathematics Meetings, Conan Chadbourne and James Mai, to name a few, have used geometric arrangements in their works. This list is meant to provide a few examples and not be exhaustive.

Figure 1: *A view of the installation of my work* One to One Hundred *at MoMath in 2023.*

One possible unifying technique for creating a thematically coherent sequence of visual enumerations is to create a factorization diagram, where the prime factorization of a number is used to recursively construct a diagram of the number. Brent Yorgey has created nice visualizations of the numbers, representing a prime number p as a ring of p dots [12]. Composite numbers are represented as functional compositions of factors. For example, the number 21 is represented a ring containing 7 clusters of dots, each cluster containing 3 dots. This results in an elegant way of easily seeing the prime factors of a number.

I have explored the concept of number in several of my artworks. *Ninety-One as Sums of Four Squares* [7] depicts the five ways one can express 91 as a sum of four squares. My work *2017 = $7^3 + 7^3 + 11^3$* [11] depicts 2017 as the three indicated cubes. My works *Braces Of Eight* [5] and *Ordinal Eight* [8] represent the number 8 as sets in the the Von-Neumann construction of the integers. I have also created artwork for the mathematical constants π [9] and e [6].

The impetus for the numbers project described in this paper arose during the creation of my artwork *Septenary Circles* [10]. This piece illustrates the 26 different Langford sequences of order 7. I was searching for a way to arrange 26 items in a visually pleasing way. I ultimately used the pattern of 26 circles shown in Figure 2.

Creative Process

While working on Septenary Circles, it became clear that arranging n items in a visually pleasing pattern is an interesting problem. I decided to explore not just factorization, but to celebrate the relationships among numbers and other properties of a given number. The mathematician G. H. Hardy wrote "A mathematician, like a painter or a poet, is a maker of patterns." [3]. The work presented here (Figures 2–5) shares my love of patterns using both art and mathematics. My goal with this ongoing series is to create patterns that entice a viewer to not only ponder the concept of number, but to see properties of numbers and their connections with other numbers.

The driving question for this project is this: given a natural number n, how can one construct a visually interesting unary representation of n? To create a unified visual theme for this project, each number is presented as a collection of n white elements using a chosen motif placed on a green background that contains a marbled Perlin noise texture. For a given number, all motif elements are of identical size and shape, but allow reflections and rotations.

My process for creating the patterns included identifying which, if any, common number sequences included a particular number using tools such as OEIS [4]. I would also look at the prime factorization and other interesting features of the number. For other numbers, a more experimental approach was taken. I often used pennies, bottle caps, glass beads, and other manipulatives to create potential patterns. I took pictures with my cell phone as a record. Once I decided on a particular pattern, I would implement it in software. The pattern frequently suggested a motif, but I often started with a circle for simplicity. A digital image was then produced.

The first half of this work, *One to One Hundred*, was included in a solo exhibition at the National Museum of Mathematics (MoMath) during the summer of 2023, shown in Figure 1. Each image was printed at 8 inches (20 cm) square and framed, then mounted on a wall roughly 50 feet (15 m) long. Composite images of these and the numbers 101–200 are shown Figures 2–5. These figures should be consulted as indicated in the following text.

A design goal was to produce patterns that are reasonable to count. For example the numbers 3 through 8 are represented using just a ring of n-gons. I wanted these to be easily counted by young children. When n is sufficiently large, for example 37, a line or ring of items is not easily counted and does not efficiently use the image space. Thus, for most other numbers, other designs were chosen to be as tightly packed as the design allows, rather than just a ring or line of elements.

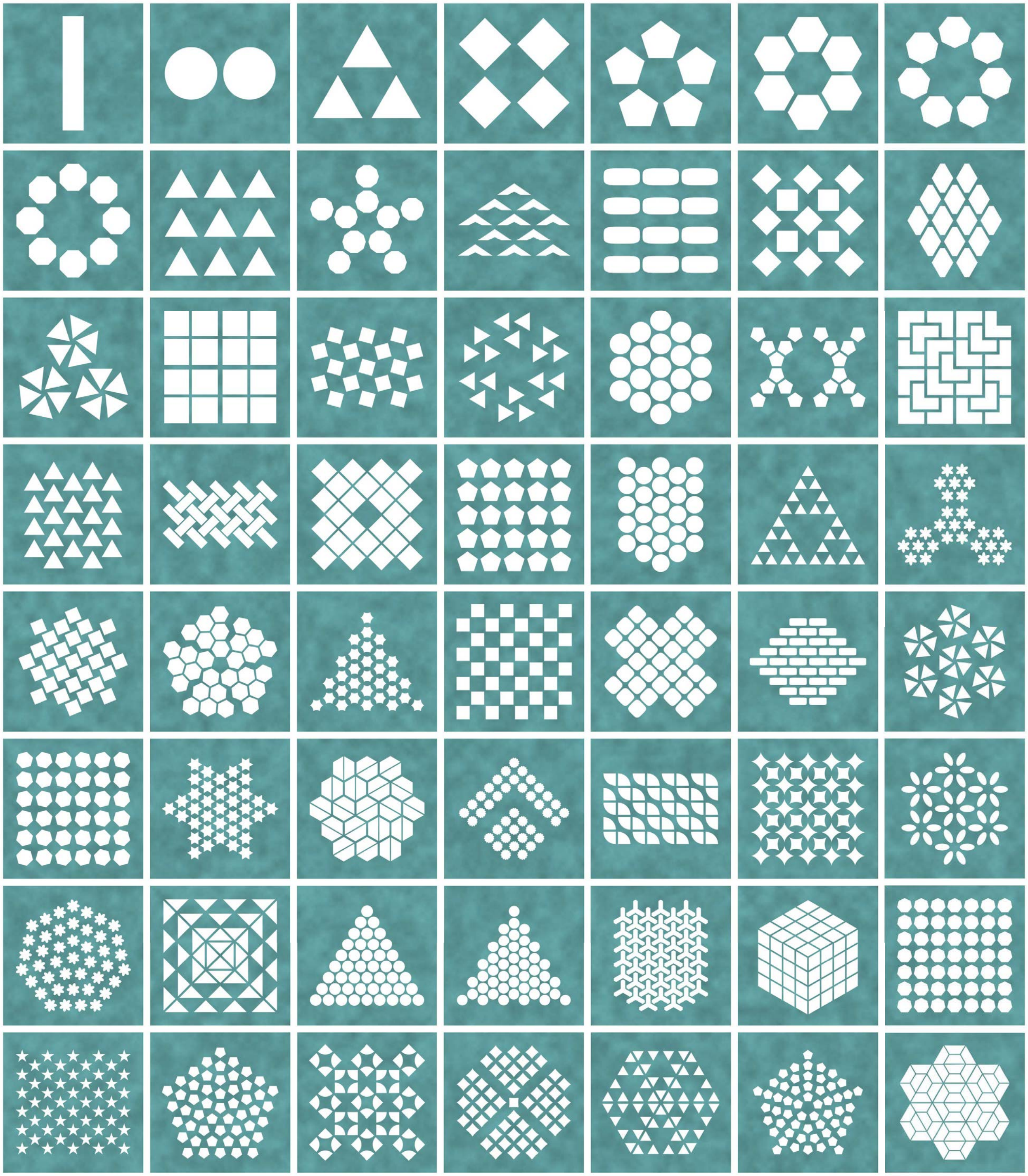

Figure 2: *The numbers 1–56 (from upper left to lower right), each as a geometric arrangement of identical visual motifs.*

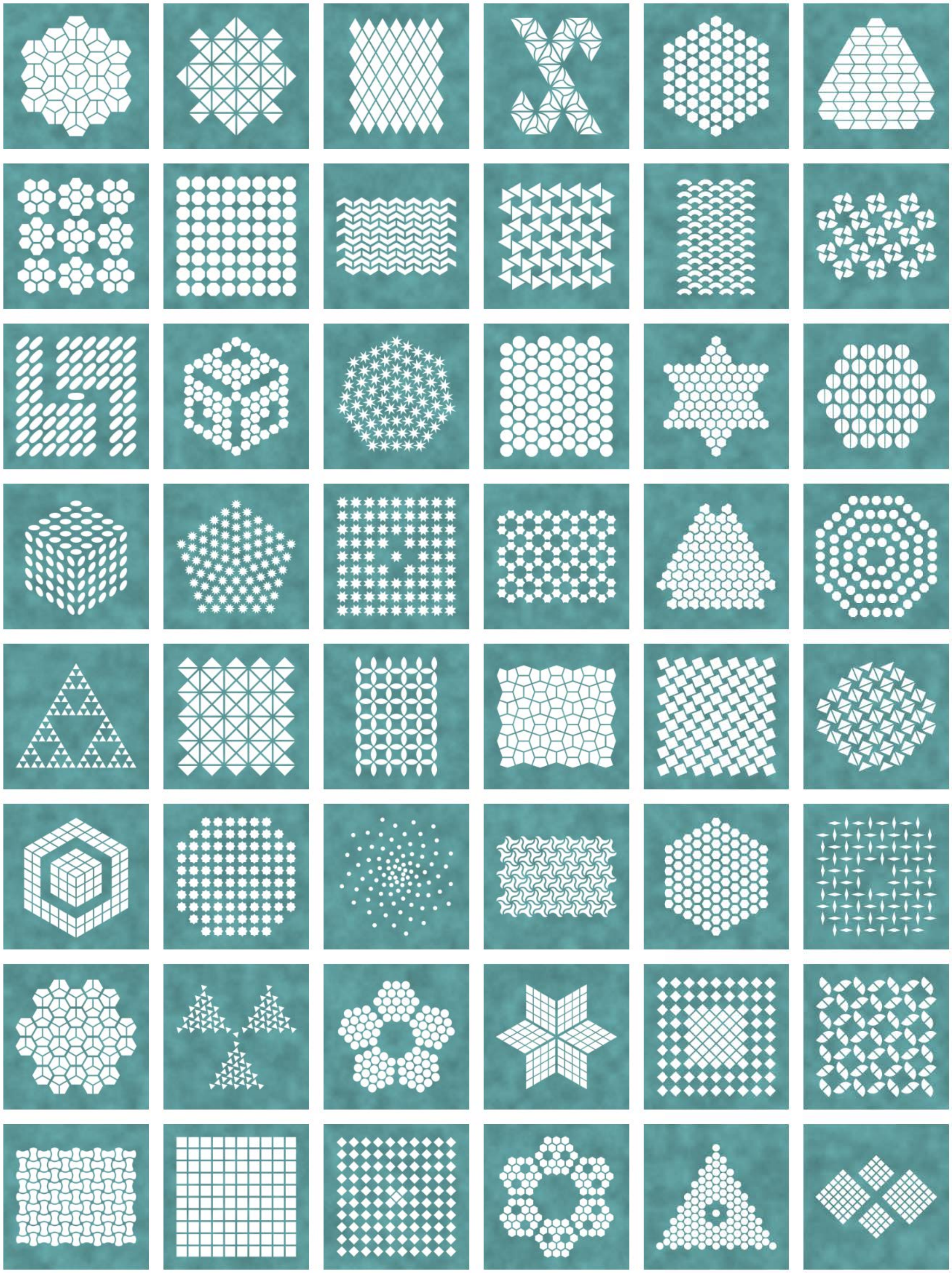

Figure 3: *The numbers 57–104 (from upper left to lower right), each as a geometric arrangement of identical visual motifs.*

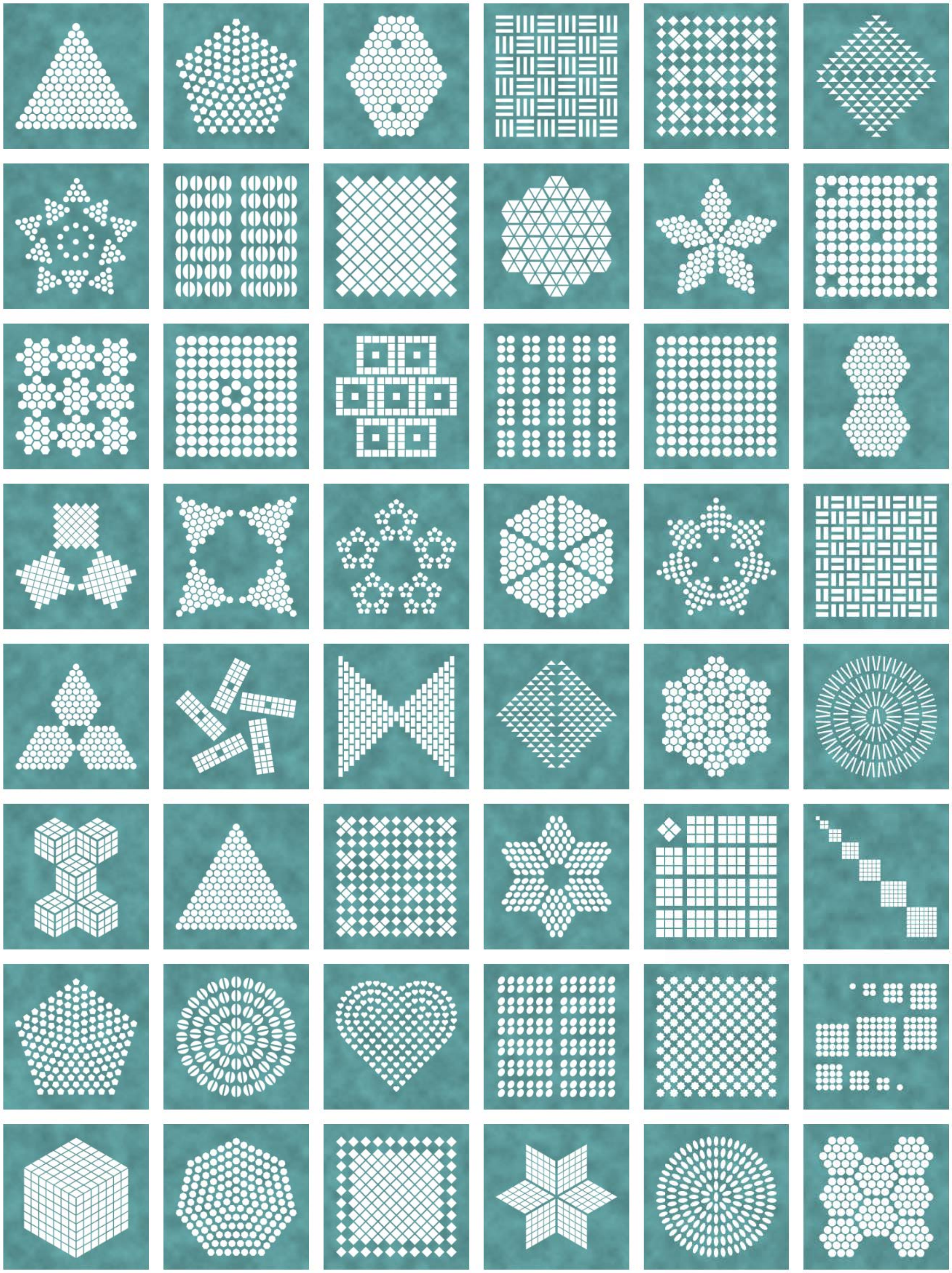

Figure 4: *The numbers 105–152 (from upper left to lower right), each as a geometric arrangement of identical visual motifs.*

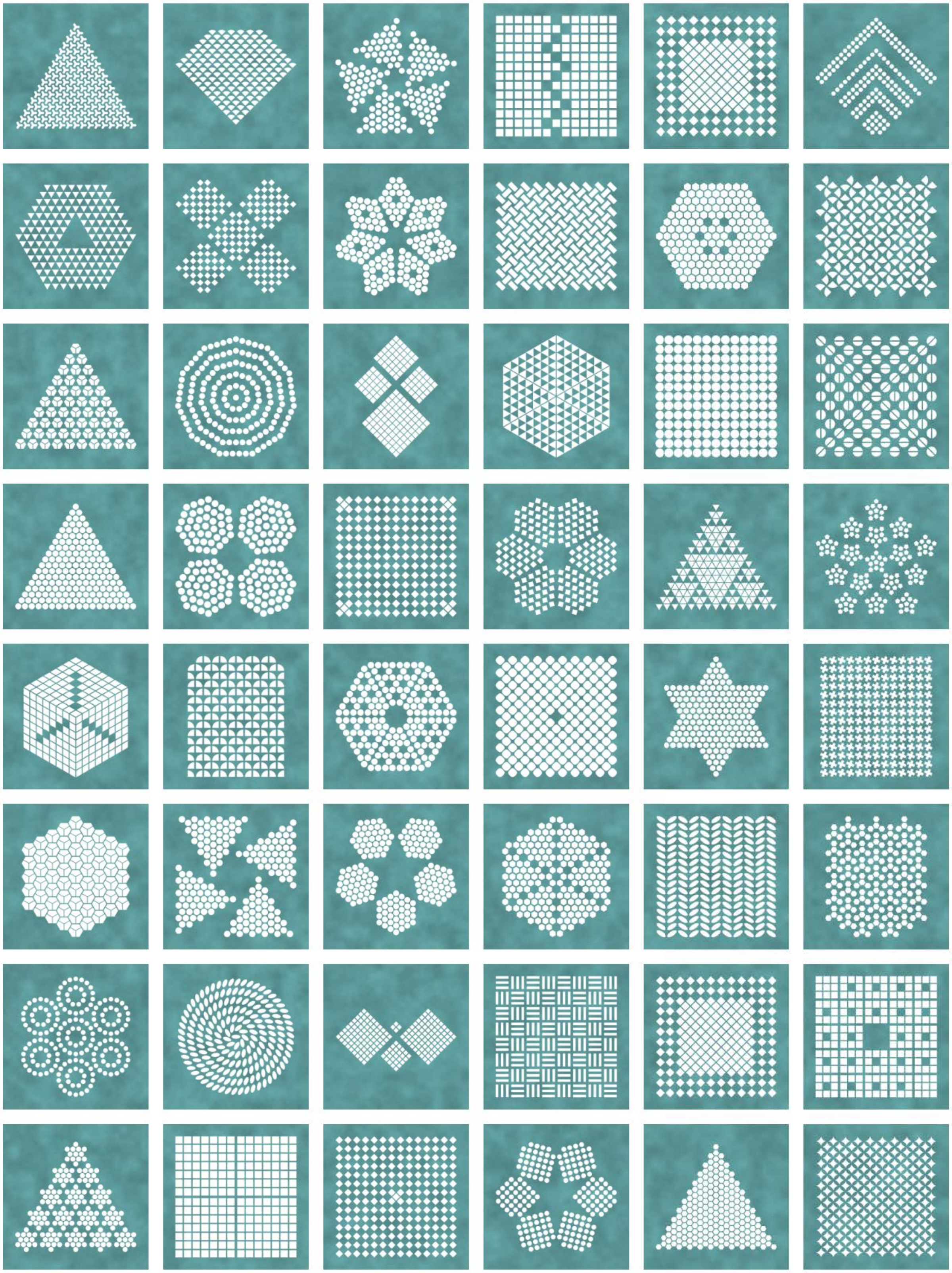

Figure 5: *The numbers 153–200 (from upper left to lower right), each as a geometric arrangement of identical visual motifs.*

Likewise, I wanted to vary the motifs to complement the pattern and add variety to the complete work. Examples of motifs used include regular polygons, circles, ellipses, superellipses, rectangles, trominoes, rhombi, and subdivisions of these. In some patterns, such as 84 and 90, other motifs from traditional tessellations were used.

Another design goal of this series was to use patterns that can be seen as an excerpt of one of the 17 wallpaper patterns and to have examples of each wallpaper pattern. Nearly all of the designs include patterns that exhibit either k-fold rotational (gyrational) symmetry about a point or k-fold mirror (kaleidoscopic) symmetry about a point, for a reasonable set of k values. The reader is encouraged to find as many different symmetry patterns as they can.

There are no hidden motifs in the designs or implied occlusion. Every pattern is strictly two dimensional even if it has the appearance of a three-dimensional isometric projection. For example, see numbers 48, 75, and 135.

Some number designs exhibit a cultural aspect of a given number. For example, the number 20 uses an arrangement that reflects the corresponding Roman numerals XX. The number 69 uses an arrangement that has the appearance of the decimal digits 6 and 9. The number 94, the atomic number of plutonium, uses a pattern similar to that found in the ionizing radiation symbol. The number 143 uses heart motifs to create a larger heart; Fred Rogers popularized this number as a code for the phrase “I love you.” The digits of 143 represent the letters in each word of that phrase.

Other number designs exhibit a more mathematical aspect of the given number. For example, 60 is the number of elements in the algebraic group representing the rotations of an icosahedron; the design for 60 shows the net of a chiral patterned icosahedron. The number 89, a Fibonacci number, is represented as a phyllotactic spiral. The nth Catalan number counts, among other things, the number of ways to properly nest n pairs of parentheses; the design of 112 illustrates the 14 ways of doing this for 4 pairs of parentheses.

Prime numbers larger than 2 are either of the form $4k + 1$ or $4k + 3$ for some integer k. The primes of the form $4k + 1$ can always be expressed as a sum of exactly two squares, a result due to Fermat. For example, $13 = 4 \times 3 + 1 = 3^2 + 2^2$. Other primes of this form include 97, 101, and 109; can you spot others? On the other hand, all numbers can always be expressed as a sum of 4 squares (Lagrange’s four-square theorem), particularly primes of the form $4k + 3$; this fact is used for the numbers 167 and 191.

Figurate number sequences were another source of inspiration. The most well known such sequence are the square numbers; each square number can be arranged to form a square pattern. This was used for most numbers of the form m^2, for example, 9, 16, and 25. Interestingly, one can apply a skew transformation to a square that results in a rhombus with 60° and 120° internal angles. Such a rhombus can be used to create patterns with 3-fold and 6-fold symmetry as seen in the patterns for $48 = 3 \times 4^2$ and $96 = 6 \times 4^2$.

Similarly, triangular numbers of the form $m(m + 1)/2$ can be expressed visually as a triangle as seen in numbers 45, 105, 136, and 171.

The centered figurate number sequences also result in patterns often meeting my design criteria. For a k-gon, these are constructed using k copies of a triangular number pattern placed around a central item, thus can be expressed as $km(m + 1)/2 + 1$. When k is 6, this results in the familiar pattern used for the numbers 19, 61, and 91. The centered triangular numbers (different from the triangular numbers) occur when k is 3, such as seen for the numbers 31 and 46. The design for 71 uses the centered heptagonal number pattern. In some cases, such as 151, the motifs are placed along concentric circles rather than strict polygons.

The star numbers are numbers of the form $6m(m - 1) + 1$ and give rise to the patterns seen for the numbers 37, 73, and 181.

Powers of numbers can be used to create fractal-like patterns. This was used for the designs for $16 = 2^4$, $27 = 3^3$, $81 = 3^4$, and $125 = 5^3$.

Not every number can be nicely represented using the patterns mentioned above. One technique for overcoming this was to remove or add one or more elements from another pattern. The design for 80 removed the central point from the octagonal number 81. The design for 55 started with the circularized centered pentagonal number, removed the central point, then added five additional points.

Another way to generate new patterns is using composition of two or more existing patterns. For example, the design for 52 is obtained by composing the pattern seen for 13 with a block of 4 quarter-circles. The design for 195 is obtained as a composition of triangular and star numbers. While this often works best when the base patterns have compatible symmetries, it can provide interesting figure-ground patterns such as seen in the design for 185.

Additional information about many of the number sequences used for this work can be found in this paper's Supplement (see the Bridges Archive) and OEIS [4].

Discussion

This work has the potential to be used in a mathematics classroom setting at many educational levels to discuss counting, number sequences, and the concept of number. This could be used for elementary students to explore counting and multiplication. Simple number talk questions such as "how many?" and "how did you count it?" can potentially foster discussions that help students improve their mathematical skills. I used selected images in my discrete mathematics course to illustrate number sequence patterns.

Another way this work can be used in a mathematics classroom is describing the concept of composition, which is often taught in a pre-calculus course and used in the context of calculus. Describing the composition of visual patterns could help students gain additional mathematical intuition.

I envision this concept could be used as a joint exercise for students to create their own patterns for some set of numbers. This could create an environment for student self-discovery of what makes a square number, triangular number, or a prime number.

My initial goal was to create a design palette for arranging n items for occasions when I needed such a pattern. I would like to explore this more by creating photos of such arrangements of physical objects.

Finally, visually representing the positive integers is truly an infinite playground. At the time of writing, I have completed representations for the numbers 1 through 200 and a few others. While it is doubtful I will achieve the scope of Opałka's work, I intend to continue this series for the foreseeable future.

References

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[12] B. Yorgey. Factorization diagrams. https://mathlesstraveled.com/2012/10/05/factorization-diagrams/.