Particle-hedra: Generating Polyhedra with Inter-Particle Forces

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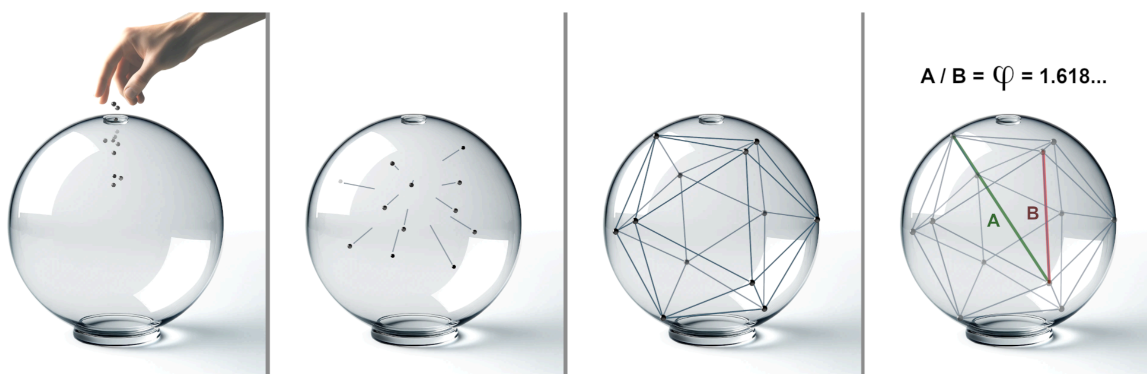
Abstract

There are many ways to create and classify polyhedra in terms of faces, edges, and vertices. This paper describes a method for generating the Platonic solids and most Archimedean solids using 3D simulated particles having attractive and repulsive forces. It could be described as a variation on the Thomson problem, referring to the distribution of points on a sphere. This method is meant to inspire creative approaches and insights on polyhedral symmetry. Particles are initialized with random positions; the forces cause the particles to converge on the vertices of polyhedra. Faces and edges are derived as secondary features as determined by the relative positions of particles. A categorization scheme is proposed, based on three polyhedral symmetry groups (tetrahedral, octahedral, and icosahedral). This could be used as the basis for an educational tool, with references to molecular modeling and dynamical systems. It also can be a way to algorithmically design a variety of 3D models.

Introduction

Imagine that you have a hollow glass sphere (Figure 1). You drop twelve black beads through a small hole at the top. You would normally expect the beads to fall to the bottom, but this sphere has two magical properties: 1. all beads inside the sphere are repelled by each other, and 2. all beads defy gravity. The repulsion forces are inversely-proportional to the distances between beads, so the closer a pair of beads, the stronger the force pushing them apart. The result is an outward expansion. Since these beads can't escape the confines of the sphere, they push up against the inside of the glass. And since there is no gravitational force, the beads can slide freely along the inner-surface until they eventually settle into a stable symmetrical pattern of mutual avoidance.



(a) (b) (c) (d)

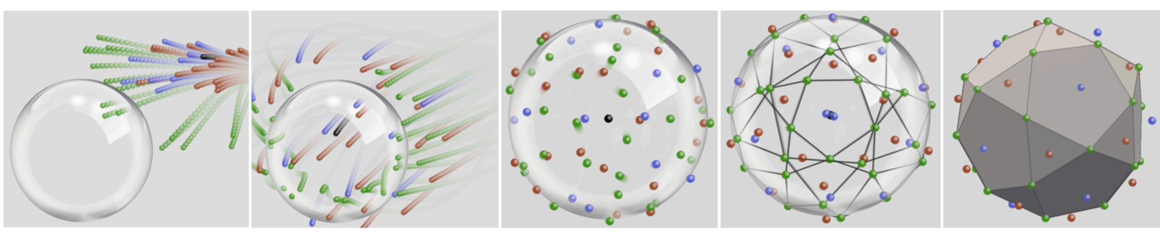
Figure 1: *Magic glass sphere: (a) dropping 12 beads into a hole, (b) there is no gravity and the beads repel each other, (c) they settle into icosahedral vertices, (d) calculating the golden ratio.*

There are 66 lines of repulsion force in all (that's the number of pairs in a set of 12). Thirty of those can be visualized as line segments of equal length: they correspond to the edges of the icosahedron, and they are the shortest (represented by length B in Figure 1(d)). The other line segments are longer and they lie inside the icosahedron; six of those have lengths equal to the diameter of the sphere, and the other thirty have an intermediate length (represented by length A in Figure 1(d)). The ratio of B to A is the golden ratio; a special feature of the family of polyhedra having icosahedral symmetry.

Consider the three types of symmetry found in the Platonic solids: tetrahedral, octahedral, and icosahedral. Dropping four beads into the sphere would result in the vertices of a tetrahedron, and dropping six beads would result in the vertices of an octahedron. And as we just saw, 12 beads results in the vertices of an icosahedron. Of the 5 Platonic solids, these are the 3 with triangular faces. The other two (the cube and the dodecahedron) are the duals of the octahedron and icosahedron, respectively. The dual of the tetrahedron is just an inverted tetrahedron. In this paper I will describe a technique that uses these 3 polyhedra as nuclei, or *seeds*, for creating 3D models of polyhedra in a way that might bring to mind molecular engineering. Specifically, this paper proposes a way to generate the cube, dodecahedron, and other symmetric polyhedra as derived from these three seed polyhedra. This is done by introducing multiple *types* of particles with custom (asymmetric) interactions.

The glass sphere I just described was only meant to serve as an introductory illustration. Instead of having beads constrained to the interior of a sphere, my technique takes a more open-ended approach: it is a simple simulation in which freely-moving particles are equally attracted to a central point, but repelled by each other in varying amounts. The attractive force is proportional to a particle's distance from the center (like the force of a rubber band). And the repulsion forces are inversely-proportional to distances between pairs of particles. These two opposing types of forces enable pockets of equilibrium, so that each particle can find a place to settle.

To give you a quick visual preview of the process, Figure 2 illustrates the creation of an icosidodecahedron, using four types of particles (black, blue, red, and green), each having different magnitudes of repulsion from the other types. Most of the particle types in this case are used as *scaffolding* to constrain the green particles, which in turn form the vertices of the icosidodecahedron. The translucent unit sphere in the illustration is only used for visual reference. There is no explicit ordering of particles; each time they are initialized, they settle into different positions, and yet the resulting polyhedron always looks the same (though arbitrarily rotated). This process is explained in more detail below. These images are taken from an interactive web application I made for this exploration [11].

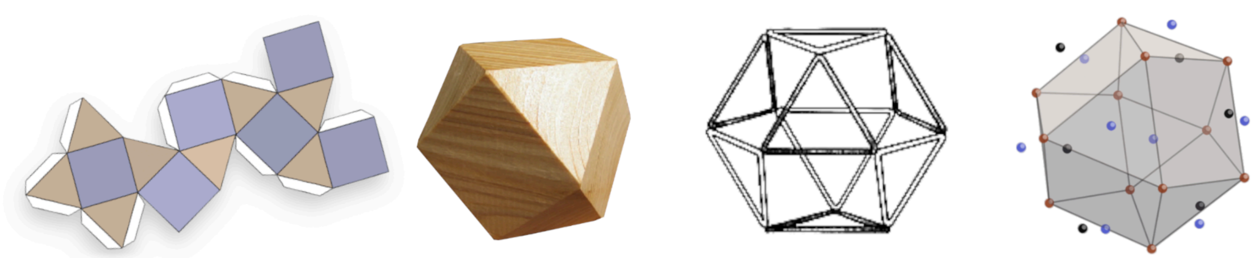


(a) (b) (c) (d) (e)

Figure 2: *Making an icosidodecahedron: (a) randomly throwing four kinds of particles into the scene, (b) particles negotiate their resting positions (translucent unit sphere shown for visual reference), (c) particles have almost settled into place, (d) identifying edges, (e) identifying faces.*

Different ways to Imagine Polyhedra

The Platonic and Archimedean solids have many fascinating properties, and these can be realized through the various ways they can be created. You may have learned how to construct a cuboctahedron by gluing together 14 cut-out cardboard pieces (6 squares and 8 triangles (Figure 3(a))).



(a) (b) (c) (d)

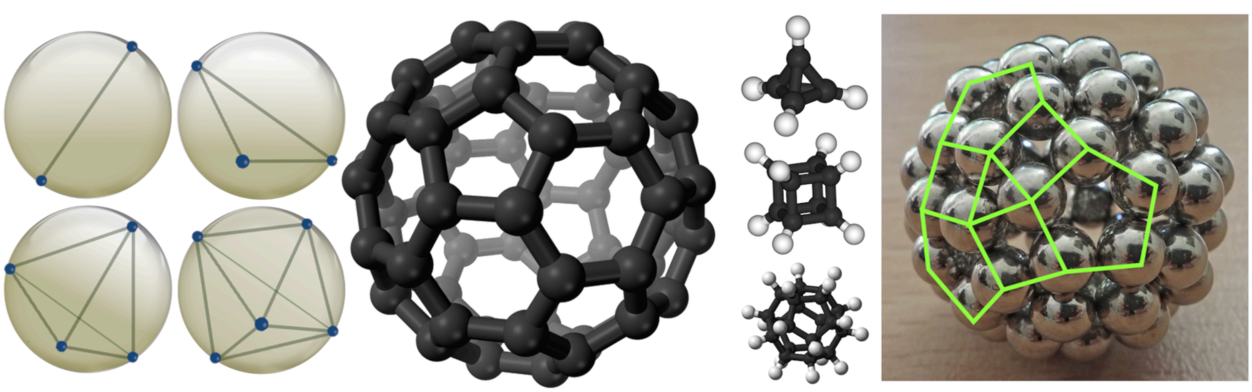
Figure 3: *Some ways to make a cuboctahedron: (a) gluing faces together, (b) truncating wood, (c) connecting sticks, (d) particle-hedron.*

Upon discovering that you could create the same shape by sawing off (*truncating*) the corners of a cube of wood (Figure 3(b)), you may come to understand the nature of this polyhedron in a more general way. You may get even closer to the Platonic ideal when you realize that this geometry can arise from a dynamical system. And with this insight, you would be in good company with Buckminster Fuller, who saw polyhedra as energy systems; as forces with constraints, and even as conceptual models ("explorations in the geometry of thinking")[4]. This is demonstrated by the *vector equilibrium* (which is like the stick-cuboctahedron in Figure 3(c) except it has 12 internal sticks radiating from the center). Without this inner-support, it becomes a flimsy cuboctahedron where the sticks can rotate at their joints. This degree of freedom allows *jitterbug* transformations whereby the symmetry-type can morph between octahedral and icosahedral using coordinated rotations of triangles. Finally (at least in the simulated world of particle physics), you can create the cuboctahedron by adjusting the attraction and repulsion forces in three kinds of particles, initializing them randomly, and watching them settle into place (Figure 3(d)).

Simulating physical phenomena with particle systems has become easier and more available in scientific modeling software and creative animation tools. The fundamental principles of particle physics can be modified and simplified to provide interactive tools to think with; to stimulate the imagination, and to learn about dynamical systems. I would even advocate strategically-chosen "cheats" against the laws of physics (e.g., allowing pairs of particles to repel each other with differing amounts of force—thereby breaking Newton's third law of motion: bodies must exert equal and opposite forces on each other). As I will demonstrate below, the ability to differentially adjust repulsion forces enables hierarchical scaffolding upon which polyhedral structures can be layered. This technique makes no reference to angles or rotations; these are considered emergent properties. The single fundamental property here is relative position, which puts this exploration into the category of *distance geometry*.

Background

The literature on polyhedral symmetry arising from dynamical systems is expansive. My interest in this area began while exploring various ways to generate regular spherical grids for running computational cellular automata that exploit the positive curvature of the sphere [10]. While working out the details of this particle system, I learned about the Thomson problem [8], conceived in 1904. And it is quite relevant here! Given N electrons that are mutually repulsive and constrained to the surface of a unit sphere, how do they align themselves to minimize energy? In the Thomson problem, the repulsion forces are given by Coulomb's inverse-square law (imagine Newton's inverse-square law of gravity…but in reverse). It turns out that the forces I am using are sufficient for creating the same basic effect (though not necessarily scaled to the unit sphere). The Thomson problem is illustrated in Figure 4(a), where 2, 3, 4 and 5 electrons form a line segment, triangle, tetrahedron, and triangular dipyramid. Some examples related to the Thomson problem are described by Paul Bourke [1] and Martin Trump [9]. Popko and Kitrick, in the book, *Divided Spheres*, describe "self-organizing grids" starting from an initial distribution of points on a sphere, then iteratively shifting their positions so as to converge on more even distribution [5].



(a) (b) (c) (d)

Figure 4: *(a) Four examples from the Thomson problem, (b) Buckminsterfullerene, (c) Platonic hydrocarbons, (d) a rhombicosidodecahedron made of magnetic spheres.*

Although we cannot experience molecules directly, we take inspiration from molecular engineering, regarding some amazing discoveries, like the Buckminsterfullerene ("buckyball") (Figure 4(b)). It has 60 carbon atoms arranged as the vertices of a truncated icosahedron. Several polyhedral hydrocarbons can be synthesized, including tetrahedrane, cubane, and dodecahedrane (Figure 4(c)).

Engineers, sculptors, and architects have taken inspiration from Buckminster Fuller, who popularized a systems-thinking approach to polyhedral geometry [4]. Caspar Schwabe compares Fuller's vector-equilibrium jitterbug with Rudolf von Laban's icosahedral-based dance notation system; both are examples of "hands-on" experience, having interactive dynamics at their core [6].

Segerman and Zwier report on several stable polyhedral structures made of magnetic spheres [7], including a rhombicosidodecahedron. This is shown in Figure 4(e) with an overlay of some of the triangular, square, and pentagonal faces of this beautiful Archimedean solid.

How it Works

A particle is defined as a point-mass with a position and a velocity. All particles are attracted to the center; the *origin* (0,0,0) with a force of $-\boldsymbol{pa}$, where $\boldsymbol{p}$ is the position of the particle and $\boldsymbol{a}$ is a constant. In all of my experiments I set $\boldsymbol{a}$ to 0.2. So, if you were to place a particle on the surface of a unit sphere centered at the origin, the particle would instantaneously experience a force towards the center with a magnitude of 0.2. To dampen oscillations and help the system settle over time, the velocity of each particle is scaled at every time step by a small friction (drag) constant $\boldsymbol{f} = 0.97$. There's nothing special about these numbers. They were chosen as workable constants; as a background upon which other numbers could be twiddled for crafting polyhedral geometry.

Let me start with the simplest possible example: a single particle initialized in a random location. After initialization, this particle quickly *rubber-bands* to the center where it comes to rest. Now let's add a second particle, along with a repulsion force determining how the two particles repel each other. This toy simulation uses a cut-off distance $\boldsymbol{m}$ (typically 2: the diameter of the unit sphere) beyond which the two particles have no effect on each other. If the distance between the particles $\boldsymbol{d}$ is less than $\boldsymbol{m}$, then there is a repulsion force equal to $(1-\boldsymbol{d}/\boldsymbol{m})\boldsymbol{r}$, where $\boldsymbol{r}$ is a constant. The particles approach the origin but this attraction is "overcome" by the proximity of the particles as they get closer. And so they gravitate to positions on opposite sides of the origin. (Figure 5(a) shows a line segment drawn between the particles for reference). Note that if attraction $\boldsymbol{a}$ were greater, the particles would get pulled-in toward the center more tightly. If $\boldsymbol{r}$ were set to a larger value, the particles would be pushed farther apart. Since $\boldsymbol{a}$ is never changed in these experiments, $\boldsymbol{r}$ and $\boldsymbol{m}$ become the two values to tweak. When $\boldsymbol{r}$ is set to 0.06 and $\boldsymbol{m}$ to 3, the length of the line segment hovers at approximately 2. Figures 5(a-c) show the 1D, 2D, and 3D simplices (line segment, triangle, and tetrahedron). Five particles create a triangular dipyramid (d); Six particles make the octahedron (e), and 7 particles make a pentagonal dypyramid (f).

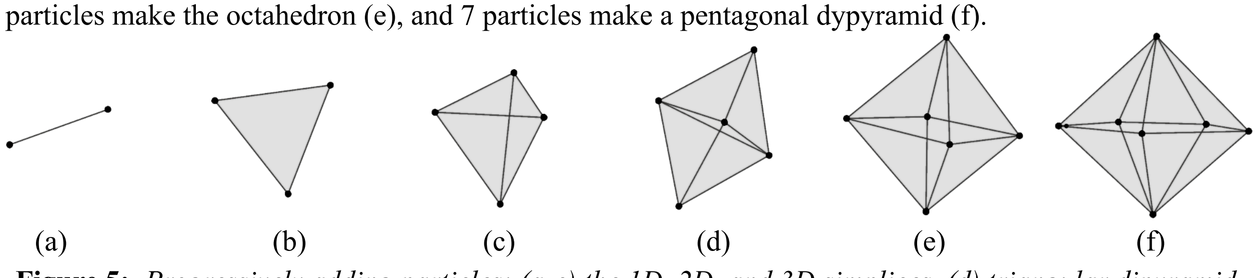
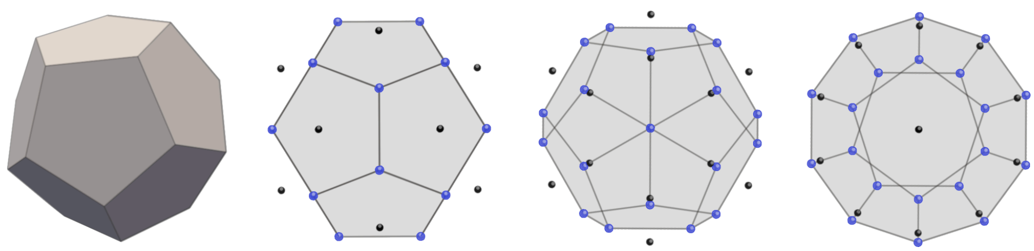


Figure 5: *Progressively adding particles: (a-c) the 1D, 2D, and 3D simplices, (d) triangular dipyramid, (e) octahedron, (f) pentagonal dipyramid.*

You may be wondering: do 8 particles make a cube? No, they do not; 8 particles make a strangely-warped square antiprism! Welcome to the Thomson problem. Moving up in particle count, we encounter the 12 particles of the icosahedron. At higher count we meet the pentakis dodecahedron with 32 particles, then progressively higher-frequency geodesic polyhedra, with 72 particles, 122 particles, and so-on. Beyond particle counts of 2, 3, 4, 6, and 12, only the geodesic polyhedra have true polyhedral symmetry. (I'm not certain if this particle system duplicates the Thomson problem, but it appears to create the same effect).

Platonic Duals as Second-Tier Polyhedra

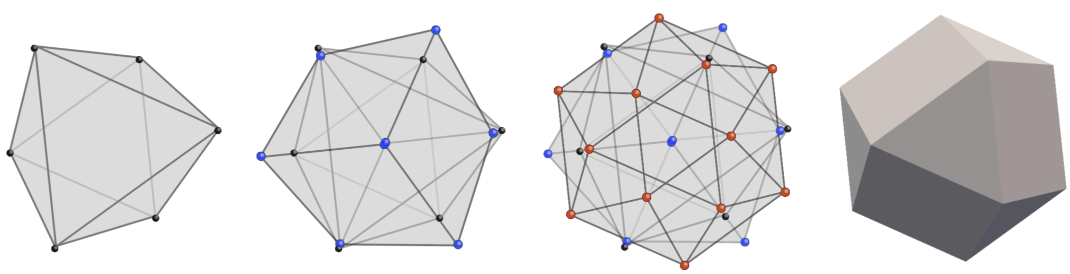
The three triangular Platonic solids constitute tier-1. The cube and dodecahedron constitute tier-2. Figure 6 shows a dodecahedron (dual of the icosahedron). The 12 icosahedral particles (black) form the nucleus, or "seed" for making the dodecahedron. Twenty blue particles are included, making a total of 32 particles. From this initial random particle soup, a dodecahedron emerges, after a bit of negotiating among the particles. The key is in the repulsions: the 20 dodecahedron particles are repelled by the 12 icosahedral particles, as well as by *their own kind*. They have more forces to contend with as they seek out places of comfort. The icosahedral particles, on the other hand, don't care about the blue particles (their associated repulsion forces are zero—a rather useful physics cheat). By not being disturbed by the blue particles, the icosahedron maintains integrity, providing reliable scaffolding for the dodecahedron to find its balance. Figure 6(b) shows an isometric view of the dodecahedron, oriented to reveal its 2-fold symmetry axis; Figure 6(c) shows its 3-fold symmetry, and Figure 6(d) shows its 5-fold symmetry.



(a) (b) (c) (d)

Figure 6: *Dodecahedron: (a) perspective shaded view, (b-d) isomorphic view with particles, oriented to reveal 2-fold, 3-fold, and 5-fold symmetry.*

The cuboctahedron is a tier-3 polyhedron. Figure 7(a) shows the 6 particles of the octahedron (black). Eight blue particles are added to make its dual, the cube. They are repelled by the black particles as well as their own kind (Figure 7(b)). Then, with these black and blue particles serving as repulsive scaffolding, 24 red particles are added; they fall into place to form the vertices of the cuboctahedron (Figure 7(c)).



(a) (b) (c) (d)

Figure 7: *Making a cuboctahedron: (a) octahedron, (b) blue particles on octahedral scaffolding make a cube, (c) red particles make a cuboctahedron (a tier-3 polyhedron), (d) shaded view.*

In total there are 5 available particle types (black, blue, red, green, and gray). Each type can be configured to have an arbitrary repulsion magnitude ***r*** and cut-off distance ***m***, with each other particle type. Most polyhedra only require a few types and forces, and for those requiring more, I have chosen to set most values to 0 to serve as hierarchical scaffolding. To make a 4-tier Archimedean solid like the rhombicuboctahedron, I found that I had to use all 5 types. The values are shown in Figure 8, color-coded according to particle type. Notice that the single particle of type 0 (black) has no repulsion values. Its purpose is to sit in the center and keep the gray particles from taking residence in the interior as they try to avoid all 50 other particles. To this end, the gray particles are given a sufficient repulsion force from the black particle. When considered in combination with the central attraction force, this creates what is effectively a spring force keeping the gray particles near the sphere. It acts as a "soft-collision" version of the hard sphere constraint associated with the Thomson problem.

type 0: num 1							center
type 1: num 6	r0=0	m0=0	r1=0.025 m1=2				octahedron
type 2: num 8	r0=0	m0=0	r1=0.007 m1=2	r2=0.01 m2=2			cube
type 3: num 12	r0=0	m0=0	r1=0.005 m1=2	r2=0.005 m2=2	r3=0.005 m3=2		cuboctahedron
type 4: num 24	r0=0.08	m0=1.2	r1=0.01 m1=0.6	r2=0.01 m2=0.6	r3=0.005 m3=0.6	r4=0.008 m4=1	

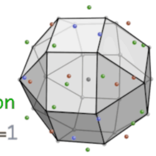


Figure 8: *Rhombicuboctahedron repulsion magnitudes (r0, r1, etc.) and cut-off distances (m0, m1, etc.) for each particle type in relation to each other type.*

Finding Edges and Faces

Given just a pile of particles, edges and faces are unknown; they must be derived from the relative positions of particles in order to properly specify and render the polyhedron. After random initialization, particles can take time to settle into place, and in the app they can be interactively jostled around at any time. Figure 9 shows a truncated dodecahedron being shaken, followed by the particles slipping back into a new stable configuration. It takes anywhere from a second (tetrahedron) to 5 seconds when there are many particles involved, due to intervening chaos. Sometimes they don't make it and slip into a metastable mutation (nothing a bit of jostling can't fix). In the app, the calculation to identify edges and faces is applied periodically (about once per second).

For the tetrahedron (a simplex) the edge/face calculation is easy. But for any other polyhedron it can get tricky. Consider the icosahedron, which presents 66 line segments (unique pairs of particles) and 220 triangles (unique triplets). Only the edges and triangles on the surface (the convex hull) are of interest. To identify edges, I first scan through every pair of particles to find the closest pair (the smallest distance). This distance represents the length of a polyhedral edge. Then I scan every pair again, this time to identify and collect the cluster of pairs whose distances are very close to the smallest distance. "Very close" is determined by a hand-tuned margin of error. The indices of these pairs of particles are then stored in an edge array, which is used for rendering.

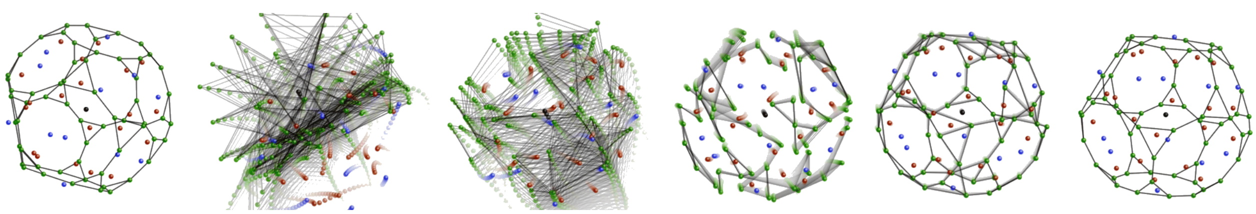


Figure 9: *Truncated dodecahedron being interactively shaken; the particles settle into a new configuration and edges are periodically re-calculated.*

Finding faces is a bit more complicated. Rather than calculate the convex hull (having triangular facets), I identify the polyhedral faces, which may have more than 3 sides. A face is defined as a set of edges that are approximately coplanar, where the plane faces "outward" (approximately perpendicular to a ray from the origin to the centroid of the edge particles), and where the plane is "outside" any other particle (i.e., it is on the convex hull). The algorithm scans through all edges for reference, and applies a search for edge groups meeting these criteria, and stores the edge indices for each identified face in clockwise.

Alternative polyhedron rendering techniques include scanning every possible triangle and building a triangle array from a subset (e.g., all triangles smaller than a given area, or within a given range). This could include, for instance, triangles inside an icosahedron to make a stellated dodecahedron. The type of particle can be specified for determining faces; the app allows switching between particle types for rendering, which can be used to view scaffolding polyhedra.

Classification

An icosidodecahedron (Figure 2(e)) can be a wonderful thing to meditate on as you roll it around in your hand. But something else happens in your mind when you see it positioned along with its kin: you see a *family.* It can be understood as the mid-state of a *morphing*—a progression of truncations, as shown in

Figure 10, row c, icosahedral column. The icosidodecahedron can be represented with 6 intersecting decagons, each lying on a plane that cuts the solid in half. It is the *rectification* of the icosahedron and the dodecahedron (the edge-*midpoint truncation* between the two). It has a partner on the octahedral family side: the cuboctahedron, shown in Figure 10, row c, octahedral column. The cuboctahedron can be represented with 4 intersecting hexagons, each lying on a plane that cuts the solid in half. Coxeter referred to these as "equatorial polygons" [3]. This is just one of the many symmetries in this family system.

What is the organizing principle behind polyhedral symmetry in the 5 Platonic solids and 13 Archimedean solids? The arrangement in Figure 10 is based on the operation of truncation. The octahedral and icosahedral columns can be seen as truncations in either direction (upward or downward). The associated changes in size that would normally result from cutting off corners is eliminated by roughly-normalizing them all in a unit sphere. I did not include the snub cube and the snub dodecahedron as I was not able to produce them given the available parameters. They are *chiral* polyhedra and I suspect they might require rotational, torque-like forces, like Fuller's jitterbug (a subject for future exploration!)

On the left side of Figure 10 is a version of the chart with standard polyhedra. On the right side is a version of the chart with snapshots of particle-hedra. These are rendered using an alternative to full faces (showing the smallest triangles—which leaves polygonal holes in faces with > 4 sides). Some of the polyhedra are missing (shown as empty spheres) because I have not yet found a good balance of forces to make them. I remain optimistic that there's a way. The 4-tier polyhedra can be identified on the left-side of Figure 10 as having some red faces. They require more tuning of forces, as demonstrated with the rhombicuboctahedron in Figure 8.

Levels of Emergence

The Conway polyhedron notation [2] refers to several operations (e.g., *dual*, *truncate*, *bevel*, etc.) that can be applied to the Platonic solids to create other solids. A somewhat different language and set of metaphors come with a particle-based system using forces. In Figure 10, arrows are overlaid to indicate the direction of hierarchy in the evolution of these polyhedra, from tier 1 to 3. Although it is not shown in the chart, the 12 particles of the truncated tetrahedron are held in place using a tetrahedron along with its dual (an inverted tetrahedron), as its repulsion scaffolding (with differing associated repulsions). So it is included as a tier-3 polyhedron.

Truncation is a *subtractive* operation. Adding a tier of particles is—in a certain different way—*additive*. Complexity grows along a different path.

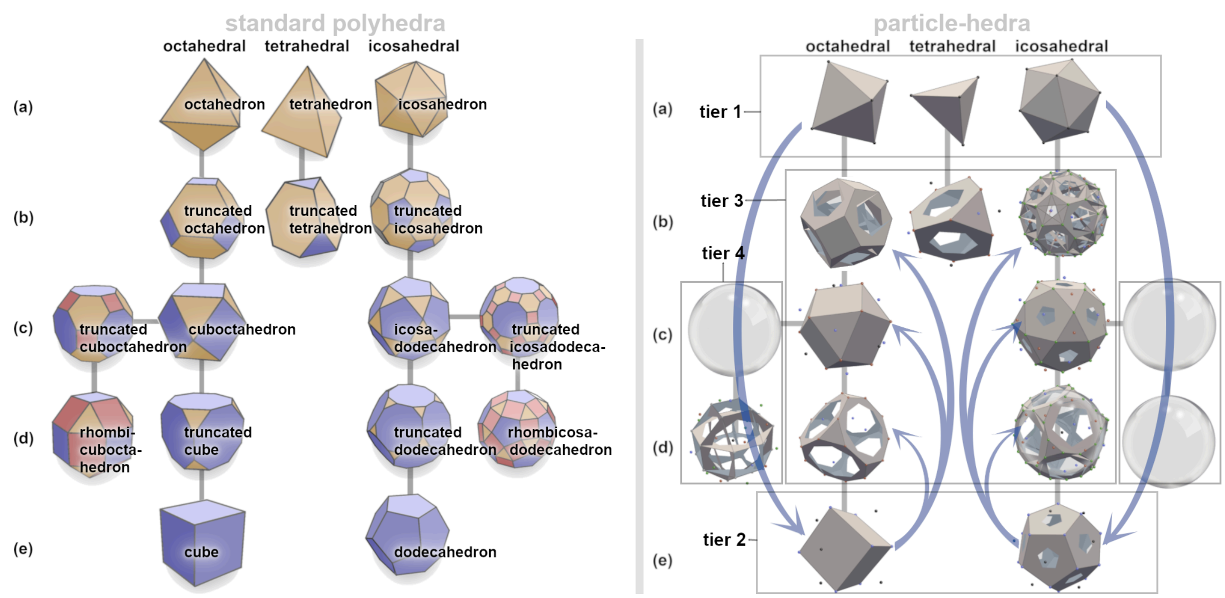
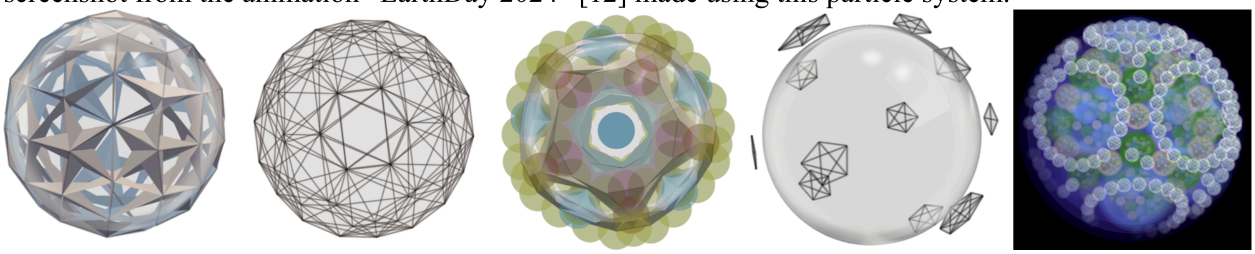


Figure 10: *An arrangement of Platonic and Archimedean solids (minus the snubs) based on truncation; Left: standard polyhedra, Right: particle-hedra, with arrows showing hierarchy.*

Concluding Remarks

A polyhedron-generating system that uses particle forces affords particular ways of thinking and creative outcomes. Figure 11 shows some variations generated with the app that reveal some peculiar properties. Figure 11(a) shows the truncated icosahedron with only the smallest-area triangles rendered, revealing sculptural relief pentagrams. Figure 11(b) shows the same polyhedron with all lines of force shown, oriented to express its 3-fold symmetry. Figure 11(c) shows the same polyhedron with particles enlarged as translucent spheres, colored according to type. Figure 11(d) shows the same polyhedron with only the shortest lines of force, revealing pentagrams located at icosahedral vertices. Figure 11(e) shows a screenshot from the animation "EarthDay 2024" [12] made using this particle system.



(a) (b) (c) (d) (e)

Figure 11: *Truncated icosahedron with glass sphere: (a) smallest triangles shown, (b) all lines of force shown, (c) particles enlarged, (d) shortest lines of force, (e) screenshot from Earth Day animation.*

This exploration is the tip of the iceberg. I haven't even covered the possibilities for stellated polyhedra, which could be made by adjusting forces so that different types of particles lie on spheres with different radii. For non-convex polyhedra, a more advanced 3D rendering environment would help, as there can be many intersecting internal polygons involved. I also haven't mentioned the possibilities with having arbitrarily-numbered particles in the different tiers, which might have hidden surprises. The experience of adjusting forces and adding particles could be further enhanced with virtual reality, enabling something that could be described as *immersive sculpting with particle force fields*.

Acknowledgements

I would like to thank Henry Segerman and George Hart for suggestions and ideas on this subject.

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