An Initial Attempt at a Mathematical Treatment of Translational Coordinate-Motion Puzzles

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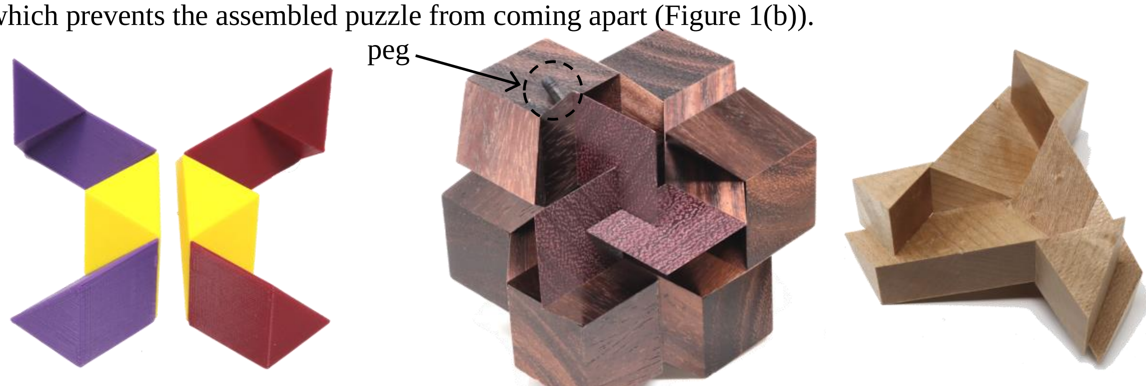
Abstract

Coordinate-motion is a geometric concept found in mechanical puzzles. In order for a coordinate-motion puzzle to come apart or disassemble, three or more pieces must move simultaneously. In this initial investigation, we present many examples of coordinate-motion puzzles and attempt to categorize their movements. To simplify our model of a coordinate-motion object, we include only translations of pieces; we do not consider piece rotations.

Introduction

Rosebud is a six-piece mechanical puzzle discovered by Stewart Coffin in 1983 [6]. It consists of three identical pieces and three mirror image pieces, shown in Figure 1(a). The pieces are based on the geometry of the rhombic dodecahedron and the assembled shape is a partial stellation of the rhombic dodecahedron.

Rosebud assembles into several symmetrical shapes, but the assembly in Figure 1(b) can only be accomplished when all six pieces translate toward a central axis simultaneously. I created a video showing the assembly and disassembly [9]. Assembly of the wood version is challenging, because one must find the starting position from which all six pieces can slide together. The starting position is unstable as no piece is supported. When the puzzle was invented, people had such difficulty with the assembly that Stewart Coffin created a special jig to hold the pieces in the starting position (Figure 1(c)), and he added a peg which prevents the assembled puzzle from coming apart (Figure 1(b)).



(a) (b) (c)

Figure 1: *(a) Two mirror image pieces of a 3D printed* Rosebud *puzzle, (b) an assembled wood* Rosebud *partially apart, (c)* Rosebud *assembly jig (wood puzzles made by Scott Peterson).*

Rosebud can be considered a geometric solid dissected into six pieces. As it comes apart each piece translates away from a central axis. The piece movements can be described using a non-negative parameter t which begins at 0 when the puzzle is assembled and ends when the pieces no longer interlock. In this sense the disassembly space of Rosebud is one-dimensional [2].

The remainder of this paper will consider objects which come apart when all pieces move simultaneously. When we refer to the *assembly* of a puzzle, we refer not to the final state but rather the movement of pieces toward that final state. Conversely, the *disassembly* refers to the opposite action—the movement of pieces resulting in the object coming apart. The objects we consider have three special properties:

1. The object can be considered a geometric solid dissected into n pieces.
2. In order to disassemble, all pieces must move simultaneously, and these movements only involve translation. Piece rotations are not allowed.
3. The result of piece displacements is that the object comes apart, i.e. as the movement is extended eventually the pieces no longer touch one another.

There are many objects composed of pieces which move simultaneously that we do not consider. These include any mechanism with gears or hinges. A good example of what I consider a hinged mechanism is the Hoberman sphere [10]. These mechanisms do not fall within our scope because they do not have the above three properties.

Here is a preview of what comes next: first, we present a mathematical theory of coordinate-motion involving only translations. Second, we give many examples. In order to gain insight, the reader may wish to skip the theory and peruse the examples.

Theory of Coordinate-Motion

In [5], Stewart Coffin defines a *coordinate-motion* puzzle as: a puzzle which, at some stage during the assembly, requires the simultaneous manipulation of three or more pieces or groups of pieces.

This definition is our starting point; our goal is to develop a mathematical theory of coordinate-motion. Coffin's definition does not rule out piece rotations; however, it is clear his definition refers to piece translations as his book does not include any examples of coordinate-motion with rotations.

The simultaneous piece translations will be represented by a set of *piece movement vectors* (PMVs), $\{\vec{p}_i, i = 1, \cdots, n\}$, where $\vec{p}_i$ is the (constant) direction of displacement of piece i. $t = 0$ defines the assembled state with the center of mass of piece i located at $\vec{x}_i$. At time t the displacement of piece i is $\vec{d}_i = t\vec{p}_i$, its center of mass is located at $\vec{x}_i + \vec{d}_i$, and its speed is $|\vec{p}_i|$.

By adding a constant vector to each $\vec{p}_i$ we can assume that the sum of $\vec{p}_i$ over all pieces is the zero vector. We then define the disassembly specified by $\vec{p}_i$ as *coordinate-motion* if the set $\{\vec{p}_i, i = 1, \cdots, n\}$ contains at least three different vectors. Note that we define coordinate-motion as a property of a disassembly, whereas Coffin defines it as a property of the puzzle or object. Our definition allows for a puzzle with a coordinate-motion disassembly as well as a different non-coordinate-motion disassembly.

The PMVs cannot all be $\vec{0}$, as this would imply that no piece moves (it is fine if some $\vec{p}_i = \vec{0}$, these pieces are not moving). The only set of PMVs which is not coordinate-motion is when each $\vec{p}_i$ is either parallel or anti-parallel to some vector $\vec{c}$. This corresponds to an object separating into two parts, and describes the case where one piece can be removed from the puzzle, as well as the case where the puzzle comes apart in halves.

One aspect of coordinate-motion which is difficult to capture in a definition is that the pieces must constrain their own movements in such a way that only the piece translations specified by the PMVs are possible. Eventually, if the piece displacements are extended far enough, pieces will no longer touch. When this happens, coordinate-motion has ended, because many additional piece movements are possible.

To this point we have considered the case where the disassembly is one dimensional, as evidenced by the single parameter t. We shall see examples where we need multiple PMVs. For these cases we have D time-like parameters $t_d \geq 0$ with the assembled state defined by $t_d = 0$. The disassembly is described by D piece movement vectors $\vec{p}_{\mathrm{d},i}$ and the displacement of piece i is

$$\vec{d}_i = \sum_{d=1}^{D} t_d \vec{p}_{d,i} \ . \qquad (1)$$

The *disassembly space* Ψ is defined by the displacements $\{\vec{d}_1, \vec{d}_2, \cdots, \vec{d}_n\} \in \mathbb{R}^{3n}$ for all $t_d \geq 0$. Ψ is a subset of $\mathbb{R}^{3n}$, each point in Ψ specifies displacements of all n pieces such that they are guaranteed not to overlap. It can also be thought of as the configuration space of the puzzle which includes the assembled state ($t_d = 0$). We can think of the disassembly of the puzzle as a path through the disassembly space from the assembled state to a point where it comes apart.

Mathematically, Ψ is a *convex cone* [11], which is defined by the following properties:

1. If $x \in \Psi$ then $\alpha x \in \Psi$ for any scalar $\alpha \geq 0$.
2. If $x \in \Psi$ and $y \in \Psi$ then $x + y \in \Psi$.

For the disassembly space, the first property specifies that if *x* specifies valid piece displacements the displacements can be extended and the pieces will not interfere or block one another. This is an assumption which must be verified for each puzzle. The second property specifies that any two displacements can be made sequentially and no blocking occurs. This is also an assumption which must be verified in each case. The two properties hold for all the examples except for the final example (Trinity).

For Rosebud, the coordinate-motion specified by the single PMV can be extended forever with no piece interference. If the peg in Figure 1(b) is added, it modifies one piece in a way that prevents the puzzle from coming apart. The peg blocks the coordinate-motion which manifests as a violation of property 1.

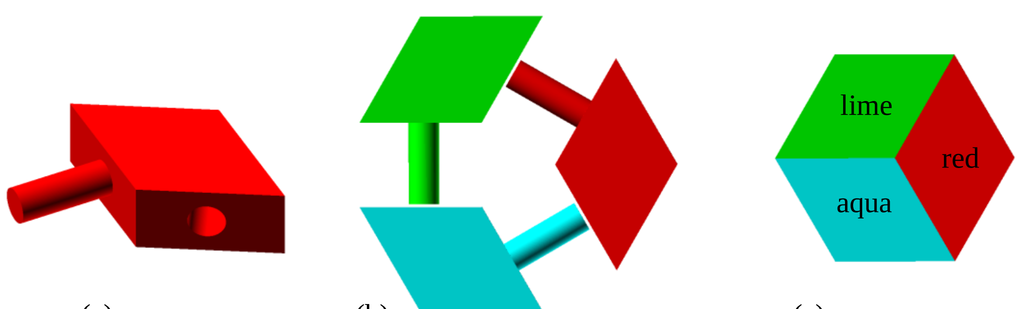
For each puzzle, our goal is to find a minimal set of PMVs $\{\vec{p}_{\mathrm{d},i}, d = 1,2,\cdots,D\}$ which capture all possible piece displacements, as given in Equation (1). We call D the *disassembly dimension*. When $D = 1$, the physical puzzle serves as a demonstration that the disassembly is 1 dimensional—no other movement is possible. When $D > 1$, the disassembly can follow an arbitrary path in $t_d \geq 0$ space. The path can be represented by D functions $t_d(s)$, each satisfies $t_d(0) = 0$ and $t_d(s) \geq 0$ (the variable s represents progress along the path). The simplest path, used in the examples, is $t_d(s) = k_d s$ where $\vec{k} = \{k_1, k_2, \cdots, k_D\}$ is a constant vector with $k_d \geq 0$. Whether a path $t_d(s)$ is coordinate-motion or not is determined by whether or not the displacements in Equation (1) involve at least three distinct vectors.

Polygon Rod and Hole Examples

We'll now demonstrate how any regular polygon or uniform polyhedron can be converted into a coordinate-motion object using the *rods and holes procedure*. To apply this procedure to a regular polygon, we first convert the polygon into a 3D object by giving it a finite thickness. We then dissect it into n pieces by cutting it into sectors of angle $2\pi/n$ and insert a rod or hole perpendicular to each interior face.

Figure 2(a) shows the result when this procedure is applied to a hexagon with $n = 3$. Three copies of this piece can be assembled into a hexagon by arranging them as in Figure 2(b) and moving them together at the same time. The PMVs for red, lime, aqua, form an equilateral triangle:

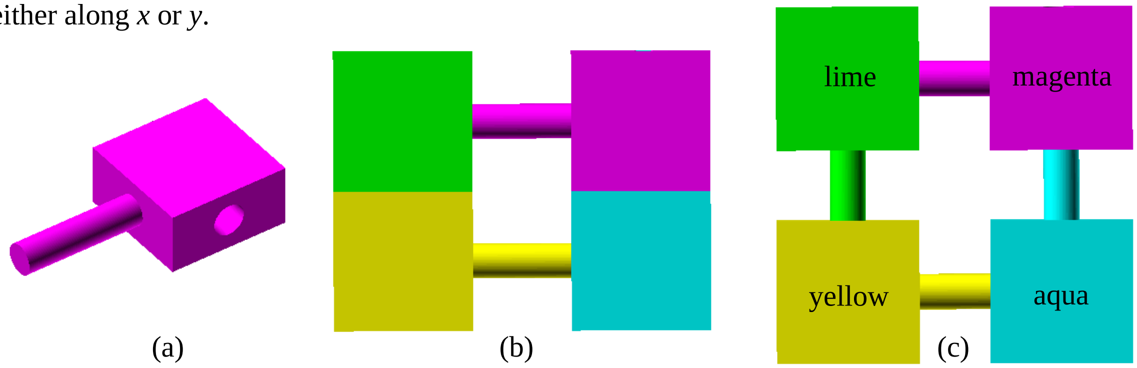
$$\vec{p} = \{(1,0,0), (cos\, 2\pi/3\, , sin\, 2\pi/3\, , 0), (cos\, 4\pi/3\, , sin\, 4\pi/3\, , 0)\}.$$



(a) (b) (c)

Figure 2: *(a) The basic piece for 3-piece coordinate-motion, (b) three pieces, aligned for assembly, (c) the assembled hexagon.*

If we begin with a square and $n = 4$, we obtain the piece in Figure 3(a), four pieces assemble into a square. An interesting twist is that the disassembly space is 2 dimensional—it can come apart in halves either along x or y.



(a) (b) (c)

Figure 3: *(a) The basic piece for 4-piece coordinate-motion, (b) coming apart in halves, (c) coming apart by 2D uniform expansion.*

The two PMVs for 1-4: magenta, lime, yellow, aqua are given by,

$\vec{p}_1 = \{(1,0,0), (-1,0,0), (-1,0,0), (1,0,0)\}$ $\vec{p}_2 = \{(0,1,0), (0,1,0), (0,-1,0), (0,-1,0)\}$.

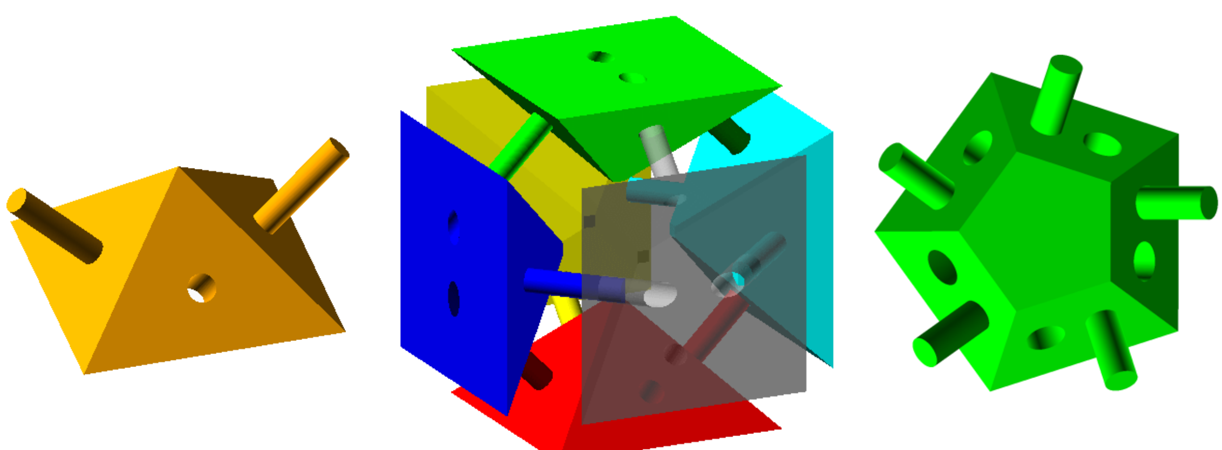
When $\vec{k} = (1,0)$, the object comes apart via $\vec{p}_1$ alone, or in halves in x. When $\vec{k} = (0,1)$ the object comes apart in halves in y. Neither is coordinate-motion. We get 2D uniform expansion in 2D when $\vec{k} = (1,1)$. Any value of $\vec{k}$ other than $(1,0)$ or $(0,1)$ is coordinate-motion, because all pieces move differently.

These two examples may seem almost trivial, but they contain many of the components of more complex 3D examples to come. The supplement contains an analysis of the case $n = 5$, including a discussion of the connection between the disassembly dimension D and the number of degrees of freedom.

Polyhedron Rod and Hole Examples

We can generate 3D examples of coordinate-motion starting with many uniform polyhedra. We first dissect the polyhedron into n identical pieces, one for each face, and then add rods and holes to the interior faces. This is the *rods and holes procedure* for a polyhedron.

For example, suppose we begin with a cube. We dissect it into six identical pieces by cutting the cube along every triangle defined by a cube edge and the center. We then add rods and holes to each interior face, as shown in Figure 4(a). The PMVs in this case are the vertices of the dual polyhedron, the octahedron.



(a) (b) (c)

Figure 4: *(a) The basic piece for the cube, (b) six pieces moving together, (c) the basic piece for a (hollow) dodecahedron.*

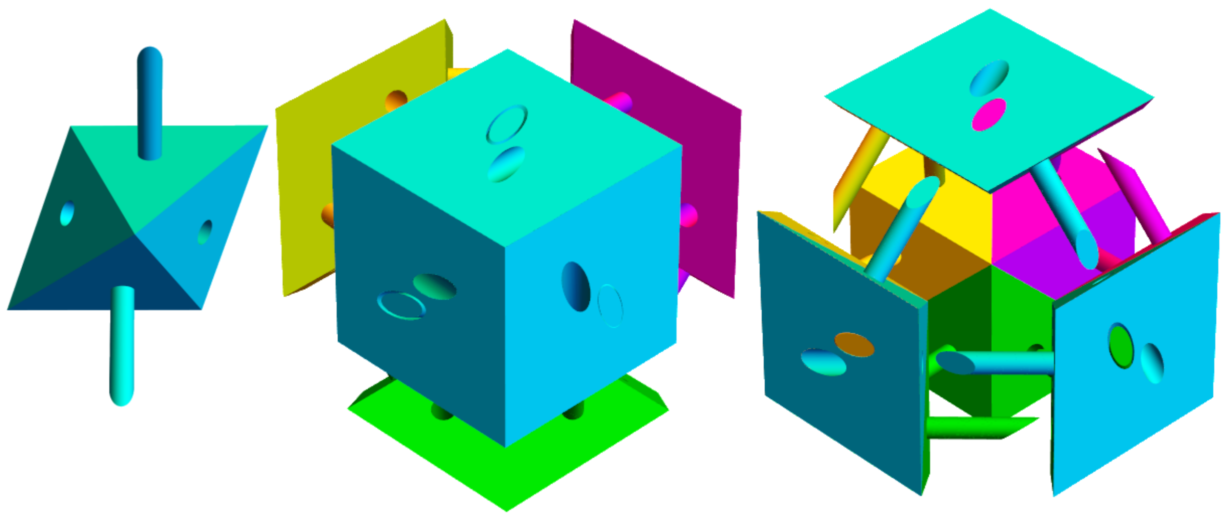
The 6-piece cube in Figure 4(b) is an example of *uniform expansion*—the pieces move outward from the center in three dimensions at constant speed. Note that the piece movement vectors alone do not determine uniform expansion. Rather, it is the PMVs together with the piece locations. Later, we'll see an example of a puzzle (Trinity) with the same PMVs where the piece locations do not give uniform expansion.

We can make coordinate-motion puzzles from the other Platonic solids using the rods and holes procedure. The procedure becomes ambiguous when the number of rods is not evenly divisible by the number of pieces. One option, which keeps all pieces identical, is to add both a rod and hole to each interior face, as shown in Figure 4(c) for the dodecahedron. Another approach is to allow the pieces to be different (adding a combinatorial aspect to the puzzle solve). Simon Dezelski [7] makes beautiful wood puzzles like this, to increase the difficulty he often adds more than one hole or rod to each interior face.

Applying the rods and holes procedure to the remaining Platonic solids gives us examples of objects with coordinate-motion disassemblies having 4 pieces (tetrahedron), 8 pieces (octahedron), 12 pieces (dodecahedron, Figure 4(c)) and 20 pieces (icosahedron). The octahedron comes apart in halves in three ways, and $D = 3$. All other Platonic solids have one disassembly dimension and the PMVs are given by the vertices of the dual polyhedron.

We conjecture that the rods and holes procedure works for any Archimedean solid, as well as any Archimedean dual. We have only verified this in the simplest cases. It is not obvious in every case that the procedure results in pieces which never overlap as they come apart.

One interesting case is the rhombic dodecahedron; Figure 5(a) shows the piece which results from the rods and holes procedure. The rhombic dodecahedron has disassembly dimension 2, as shown in Figures 5(b) and 5(c). The PMVs for t_1 are the vertices of a tetrahedron, while for t_2 the PMVs are the vertices of the dual tetrahedron. Along the diagonal $t_1 = t_2$ we have uniform expansion along the vertices of a cuboctahedron (the dual of the rhombic dodecahedron). In addition, all disassemblies are coordinate-motion (it does not come apart in halves). The rhombic dodecahedron is the only polyhedron we have found which has $D > 1$ and all coordinate-motion disassemblies.



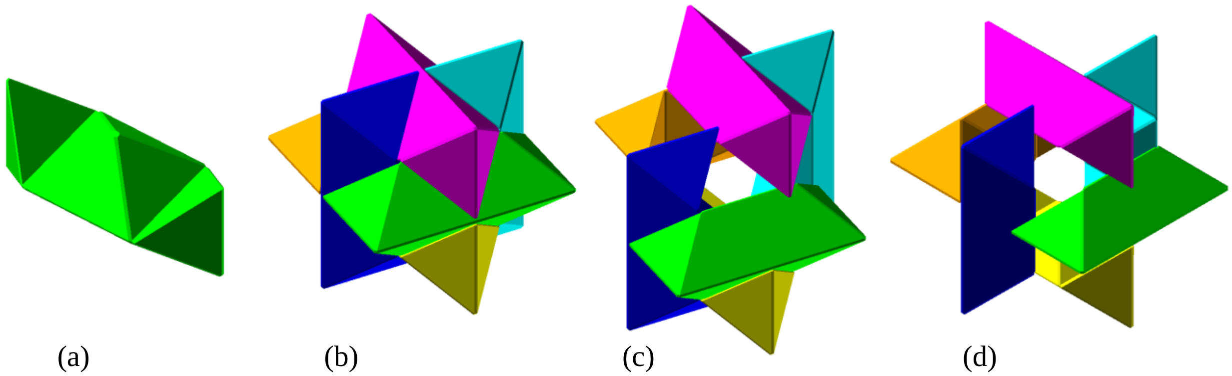
(a) (b) (c)

Figure 5: *(a) The basic piece for the rhombic dodecahedron, (b) twelve pieces moving apart via t_1,and (c) via t_2.*

Puzzle Examples

The rods and holes clearly enforce the constraint that pieces must move apart linearly. However, even without rods and holes, the piece shapes themselves can enforce a linear movement constraint.

A common puzzle is the *Diagonal Burr* [4]. This puzzle has six identical pieces, the piece shape is shown in Figure 6(a). Six pieces assemble into the first stellation of the rhombic dodecahedron (Figure 6(b)). This puzzle is not known as a coordinate-motion puzzle because it comes apart easily in halves, as shown in Figure 6(c). However, it can come apart in halves along four different axes corresponding to the four 3D cube diagonals. The Diagonal Burr has 4 disassembly dimensions, when $\vec{k} = (1,1,1,1)$ (Figure 6(d)) the pieces expand uniformly toward the vertices of an octahedron.

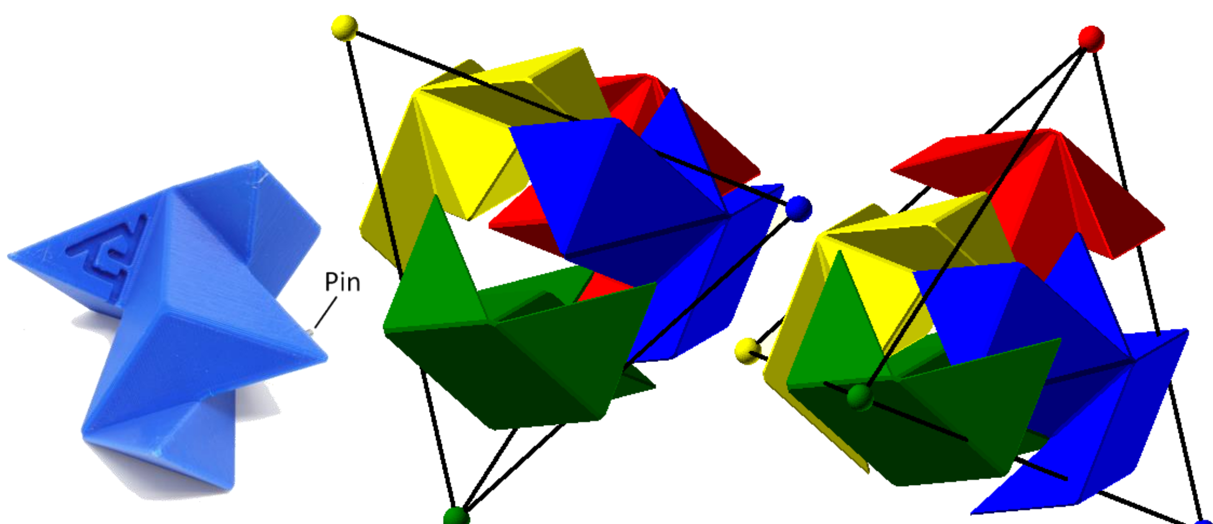


(a) (b) (c) (d)

Figure 6: *(a) Diagonal Burr piece (b) assembled (c) coming apart in halves (d) uniform expansion.*

The *Maze Pennyhedron* puzzle consists of two copies of the piece in Figure 7(a) (blue and yellow), plus two copies of the mirror image piece (red and green). The maze and pin in Figure 7(a) have been removed in Figures 7(b) and 7(c) to demonstrate the coordinate-motion. The assembled shape is a rhombic dodecahedron. This is the simplest puzzle I have found (fewest number of pieces) where $D = 2$ and all disassemblies are coordinate-motion (it does not come apart in halves). The PMVs are the tetrahedron and its dual, shown in Figures 7(b) and 7(c). Each piece is moving towards the tetrahedron vertex of the same color. If $t_1 = t_2$, or $\vec{k} = (1,1)$ or we get uniform expansion in the *x*-*y* plane (not 3D).

In the Maze Pennyhedron, certain faces are touching whenever $t_1 \geq 0$ or $t_2 \geq 0$ (and the puzzle has not come apart). If we add a pin to one face, and a 2D maze to the other (as in Figure 7(a)), we can require that the solver find a path through this maze. The disassembly can be thought of as a coordinate-motion path through the maze in (t_1, t_2) space from $(0,0)$ to the disassembled position.



(a) (b) (c)

Figure 7: *(a) Maze Pennyhedron piece (b) coming apart via t_1 and (c) t_2.*

We now return to the Rosebud puzzle in Figure 1. The disassembly of Rosebud is 1-dimensional and it uses PMVs different from any seen previously. Rosebud has one axis of 3-fold symmetry, Figure 8 shows the disassembly where this axis is aligned with the 3D diagonal of a cube. Each piece is moving toward the cube vertex with the same color. Thus, the PMVs are the cube vertices, with the two vertices along the axis of symmetry removed.

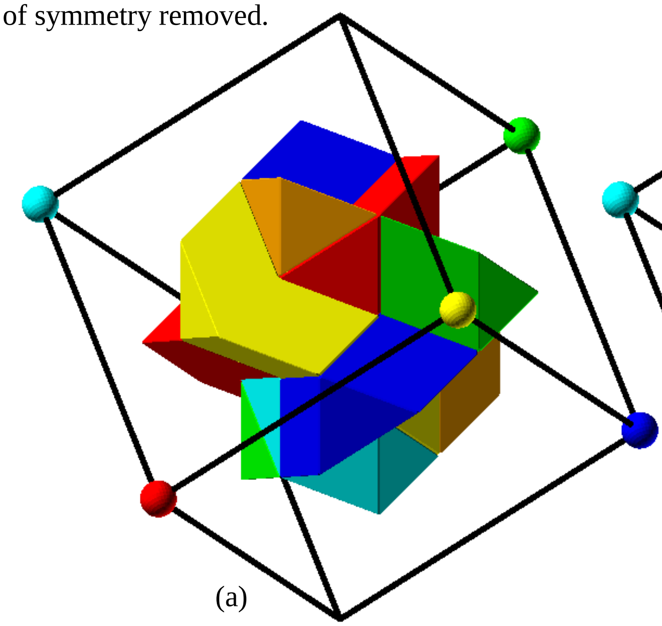


Figure 8: *(a) Rosebud assembled, $\tau = 5$, (b) Rosebud coming apart, $\tau = 4$.*

Consider the integer valued function $\tau_i(t)$, defined as the number of pieces which touch piece i at time t. For Rosebud, τ begins at 5 and decreases to 0 (by symmetry τ_i is the same for all i so we can drop the subscript). The touching pieces result in constraints on the movement of piece i. When τ decreases, a movement constraint has disappeared. The critical time t_c for Rosebud occurs when τ drops from 4 to 2. At this point new piece movements are possible (both translations and rotations) and the physical puzzle is observed to fall apart.

Trinity is an interesting coordinate-motion puzzle which was patented by Lynn Yarbrough in 2001 [8]. Trinity consists of six identical pieces—one piece is shown in Figure 9(a). The assembled puzzle is shown in Figure 9(b) with partial disassembly in Figure 9(c).

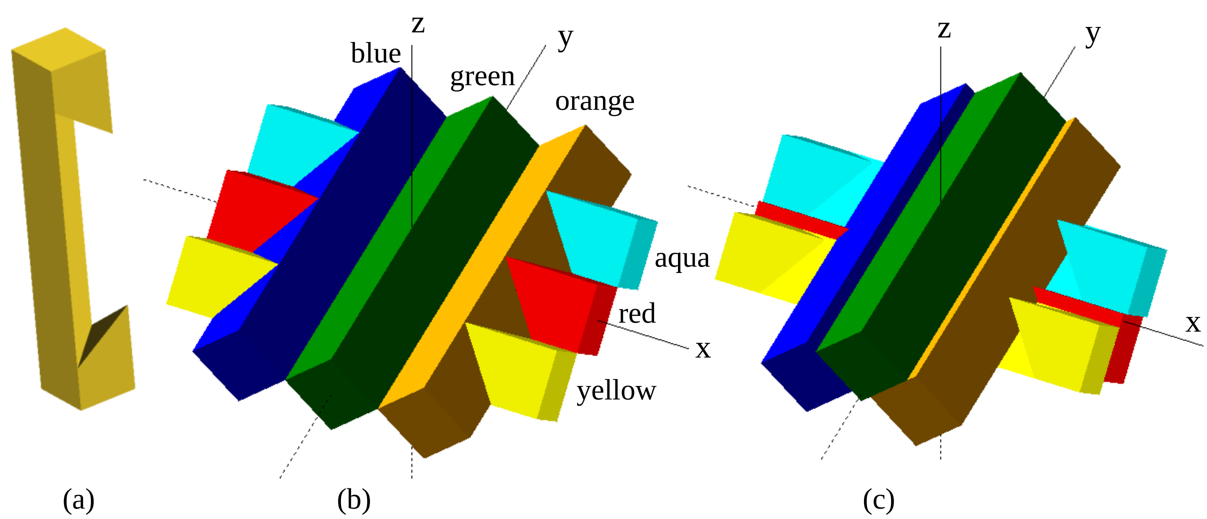


Figure 9: *(a) Trinity piece, (b) Trinity assembled, (c) coming apart via the basic PMV.*

If the pieces 1-6 are yellow, red, aqua, blue, green, orange, then the basic PMV is the vertices of an octahedron: $\{(0,1,0),(0,0,-1),(0,-1,0),(1,0,0),(0,0,1),(-1,0,0)\}$. Notice that the yellow and aqua, and the blue and orange pieces, are moving toward one another. If the disassembly continues past Figure 9(c) the yellow and aqua pieces collide; at this point the pieces are easily separated.

In Figure 9(c) the aqua and yellow pieces are free to move slightly in x, and the blue and orange pieces in y. A set of PMVs to capture all possible disassemblies is $\{(\pm 1,1,0),(0,0,-1),(\pm 1,-1,0),(1,\pm 1,0),(0,0,1),(-1,\pm 1,0)\}$. Considering all possible signs gives $D=16$, and the basic PMV can be obtained by adding them all together. This puzzle also provides an example where all the pieces do not move at the same speed.

Summary and Conclusion

One can think of coordinate-motion as a disassembly technique for certain geometric solids with a very special piece decomposition. The key indicator of coordinate-motion is that all pieces move simultaneously, and we do not allow rotations, only translations. The simplest case is where the pieces expand uniformly, translating from the center toward the vertices of the dual polyhedron. Coordinate-motion is of interest primarily when no simpler method of disassembly exists.

The most difficult coordinate-motion puzzles have one disassembly dimension ($D=1$), because they come apart in only one way. When $D>1$, disassembly tends to be easier, and assembly does not require such precise piece alignment. I have found that puzzles with $D>1$ are still quite useful in the design of coordinate-motion puzzles. One may be able to extend the pieces in a way that blocks one of the PMVs, the result is an improved puzzle with $D=1$. The Maze Pennyhedron provides a second example of how a puzzle can be modified to reduce D from 2 to 1.

Designing coordinate-motion puzzles is quite different from puzzles which come apart one piece at a time, because one must take a wholistic view starting from the symmetry of the final shape [3]. Coordinate-motion is not a common phenomenon, even among puzzles, but I suspect a targeted search might uncover many more examples.

Acknowledgements

I would like to thank George Hart for many useful discussions on coordinate-motion and Stewart Coffin for the inspiration and freely sharing of his puzzle designs. I also thank the anonymous referees, who pointed out several weak or confusing definitions. I rewrote the Theory section from scratch and added the supplement, which I hope significantly improved the final manuscript.

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