Counting a Class of Photogenic Knots on 9×9×9 Rubik's Cubes

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Abstract

In this paper, we introduce and explore a class of knots on 9×9×9 Rubik's cubes that begin from the same edge permutation and follow the photogenic knot heuristics introduced in our prior work. We identify structure in the composition of the knots, specifically the possible combinations of thread crossings on a face, and develop a labeling system to help name a knot around the entire cube. However, this introduces multiple possible labels for the same knot, up to rotation of the cube. So, we then identify a method that leverages group actions to generate equivalence classes for knots (distinct orbits under the group action). We then identify 224 distinct orbits, which demonstrates that there are 224 distinct knots.

Introduction

This work continues the first author's "Photogenic Knots on n×n×n Rubik's cubes" work in which we use traditional puzzle moves to embed mathematical knots on Rubik's cubes. The Rubik's cubes you see in this paper can be viewed as alternate solution states of the cube. That is, it is possible to re-configure the cube from the "solved" state (all faces a solid color) to states depicting knots without dis-assembling the puzzle. Photogenic knots [2] are knot projections on pixel meshes that follow 6 heuristics (Figure 1). The class of knots we explore in this paper are all photogenic. Our exploration begins with a seemingly straightforward question that escalates to requiring substantial structure and upper-level mathematics to reason about the counting problem. The work in this paper is the basis for the second and third coauthors' senior capstone projects in their respective bachelor's degrees in mathematics.

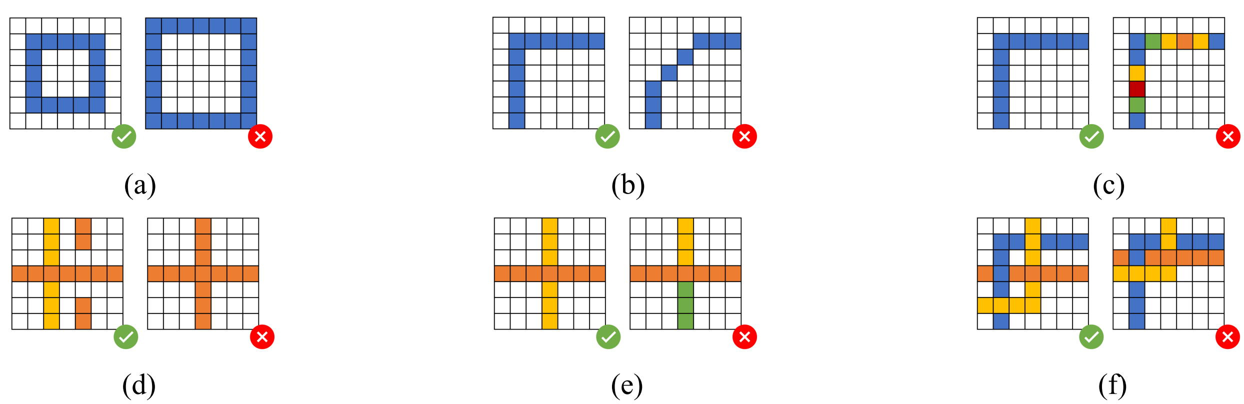
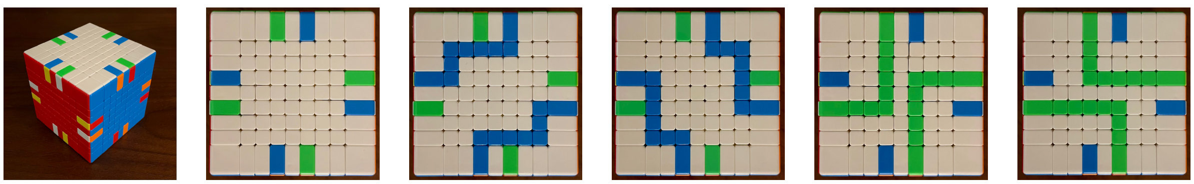


Figure 1: *Illustrating six criteria for photogenic knots with examples and non-examples: (a) knots contrast with background, (b) threads turn at right angles, (c) threads maintain color on a face, (d) discernible over/under crossings, (e) color persists after crossings, and (f) gap-i-ness reduces confusion.*

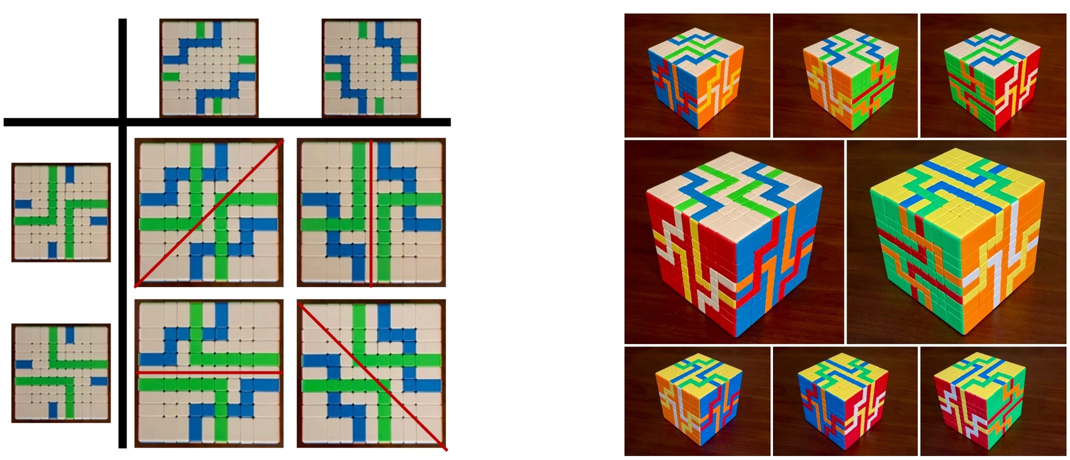
This class of patterns begins with an edge permutation on the 9×9×9 Rubik's cube (Figure 2(a) and 2(b), next page). This permutation of the puzzle's pieces can be reached via normal moves and leverages the symmetry of the cube so that each face has a solid background and eight edge pieces (four pairs) that contrast with the background. The contrasting edge pieces are colors from opposite faces of the cube (either blue/green, red/orange, or white/yellow), and the colors alternate around the perimeter of each face. These edge pieces establish the thread colors on each face. Based on the photogenic knot heuristics cited above, threads that connect the two blue edge pieces (for instance) should consist of blue pixels that turn at right angles and do not cross another blue thread. Accordingly, the thread from each blue pixel should connect to one of the blue pixels on an adjacent edge of the face. This produces two possible pairings for the blue pixels (Figures 2(c) and 2(d)). Similarly, there are two possible pairings for the green pixels (Figures 2(e) and 2(f)). To satisfy the gap-i-ness heuristic, threads of one color should trace around the perimeter of this 7×7 center pixel grid (Figures 2(c) and 2(d)), and the other color's threads should trace through the middle 3×3 grid in the center (Figures 2(e) and 2(f)). Without loss of generality, we have chosen the outer threads on the red, white, and blue faces to be the white, blue, and red threads, respectively. Similarly, the outer threads on the orange, yellow, and green faces are yellow, green, and orange, respectively.



(a) (b) (c) (d) (e) (f)

Figure 2: *(a & b) The initial edge permutation and (c – f) four possibilities for connecting edges.*

From this, on each face, there are two possibilities for the outer color's threads and two for the inner color's threads. The choice for each color is independent, producing four possible combinations between the two thread colors (Figure 3(a)). Further, the combination of thread on each of the six faces is independent of the pattern on every other face. Accordingly, there are $4^6 = 4{,}096$ possible combinations of these configurations around the six faces of the cube. In this paper, we call a combination of these configurations around the cube a ***pattern***. Figure 3(b) shows one possible pattern from different perspectives. Due to the cube's symmetry, some of the 4,096 patterns are the same after rotation of the cube. This brings us to the main research question for this paper: "*How many of these 4,096 patterns are distinct, up to rotation?*"



(a) (b)

Figure 3: *(a) Matrix of four possible thread combinations on a face and (b) an example of a pattern.*

Analysis of the Patterns

To answer our research question, we began analyzing the four possible thread combinations and realized that each combination bisects the face of the cube – one along each diagonal of the square face, one vertical bisection, and one horizontal bisection (Figure 3(a), red lines). After realizing this, we began to use the bisections to identify each combination of threads and devised a naming system to refer to these. It is necessary to identify an orientation for these bisections on each face of the cube. In our naming system, we orient the cube with one face up and a corner pointing toward the viewer. From this orientation, we chose the naming system in Figure 4 because it leverages two symmetries of the cube (discussed below).

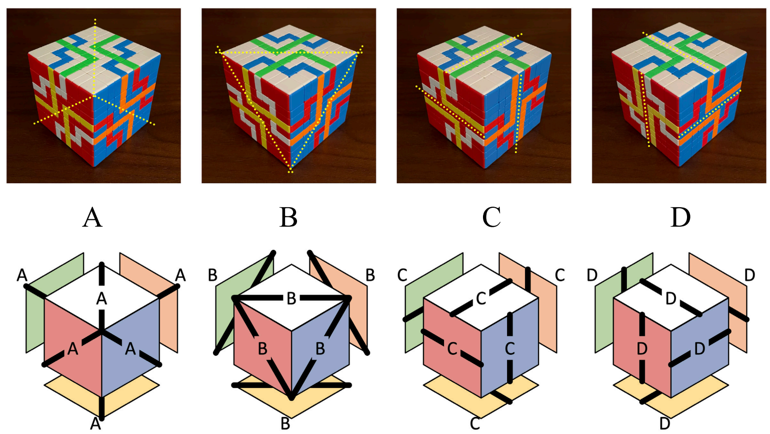


Figure 4: *Identifying the bisection of a face by the four thread combinations.*

Given any pattern from this class, we can identify the corresponding bisections and label each bisection either A, B, C, or D based on the orientations identified in Figure 4. In this paper, we call a collection of 6 bisections corresponding to a given pattern a ***cordon***. We chose this name because the bisections act like boundaries on the knot threads, pushing them to one side of a face or the other, and dictating the resulting knot pattern. Accordingly, our research question is equivalent to counting the number of distinct cordons, up to rotation of the cube. From a given orientation, we name each cordon with a 6-letter word based on naming the bisection on each face in the following order: top (Figure 5(a), 1), front-left (2), front-right (3), bottom (4), back-left (5), and back-right (6). For example, Figure 5(b) shows a cordon in which the top face is bisected along the diagonal we named A, the front-left face along the diagonal we labeled B, and so on in the order shown in Figure 5(a). Accordingly, we label this cordon in this orientation "ABCDDD".

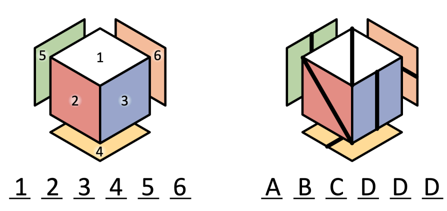


Figure 5: *(a) The order for naming bisections in a cordon and (b) example of labeling a cordon.*

As with the original patterns, rotating the cube changes the orientations of the bisections. Because a cube has 24 orientations, a given cordon might have up to 24 distinct labelings, depending on the orientation of the cube. The example above is one such cordon that produces 24 distinct labelings when rotated. We explored the different orientations of this pattern (and other patterns) and observed structures that emerged when rotating the cube into different orientations and notice how this affected the labeling of the cordon (Figure 6). We have organized the various orientations of the example pattern in Figure 6 to illustrate structures we identified in the results of our activity. Notably, our naming system is conducive to re-labeling a cordon when the orientation has the same corner in the center. This is demonstrated by the first three labelings in Figure 6. Because the red/white/blue corner remains in the center, these labelings use the same letters, rearranged in a slightly different order. The same can be said of the last three labelings. Further, the fourth labeling can be generated by swapping the first three letters with the last three letters. As we discuss below, these patterns are not coincidental, but the result of the effect that rotations have on the labels.

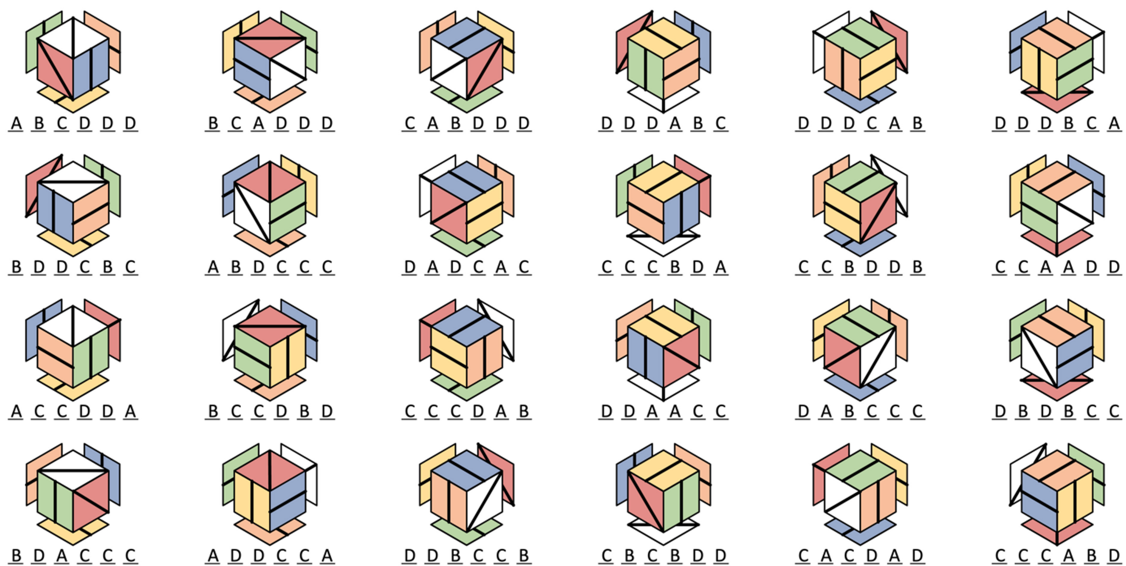


Figure 6: *24 orientations of an example cordon and their labelings.*

The patterns we identify in the labelings of the same cordon above are a natural consequence of how cube rotations interact with our naming system. Because our labels for A, B, C, and D are symmetric around the center corner and its opposite corner (Figure 4), rotating the cube 120° around an axis through those corners results in a permutation of the labels on the front half of the cube in a 3-cycle (and also a 3-cycle of the back labels) without changing the labels themselves. Figure 7 demonstrates the effect this rotation, which we call D_{120}, has on the order in which faces are listed in the labeling. So, this rotation causes the second letter of the word to move to the first position, the third letter to move to the second position, and the first letter to move to the third position. In a similar 3-cycle, the last letter moves to the fourth position, the fourth letter moves to the fifth position, and the fifth letter moves to the last position.

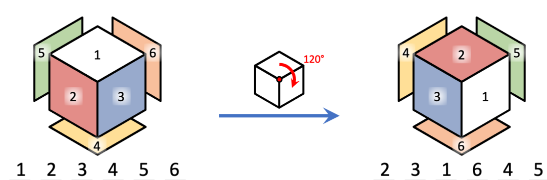
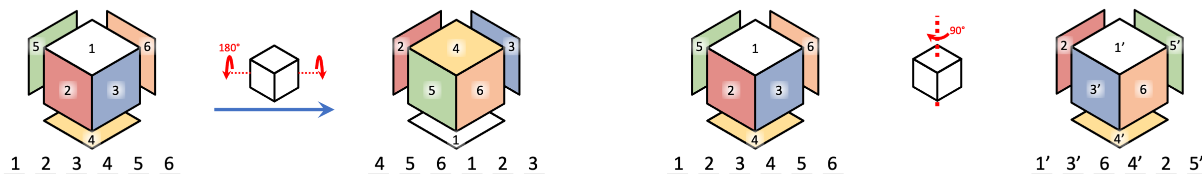


Figure 7: *Illustrating the effect D_{120} has on the labeling of a cordon.*

We also identified another rotation of the cube that results in a convenient permutation of the letters in a given labeling. Specifically, rotating the cube so that the front and back halves are swapped, and the top face goes to the bottom of the cube, causes the first three letters and last three letters to switch places (Figure 8(a)). The third rotation of the cube that we used in our analysis of these patterns is a 90° rotation about a vertical axis through the center of the top face, which we call R_{90} (Figure 8(b)). This rotation is the most complicated of these three rotations because some of the names of bisections in our labeling change orientation according to our naming system. Specifically, the top and bottom faces each stay in the same position on the cube, but turn 90°, so that a label of A on either of these faces before the rotation changes to B and vice versa (similarly for C and D). Further, the four faces around the side of the cube move in a 4-cycle so that the front-left face moves to the back-left, the back-left moves to the back-right, the back-right moves to the front-right, and the front-right moves to the front-left. To further complicate this process, two of these faces keep the same label and the other two change orientation (as with the top and bottom faces, A changes to B and vice versa or C changes to D and vice versa). We developed a shorthand for notating all the faces that change orientation by including an apostrophe on those items (Figure 8b).



(a) (b)

Figure 8: *Illustrating the effect X_{180} and R_{90} have on the labeling of a cordon.*

Together, these three rotations (D_{120}, X_{180}, and R_{90}) generate the group of rotations of a cube, so we can use them in combination to generate all 24 orientations of the cube. Figure 9 shows the effect of all 24 rotations on the generic labeling "123456". The first row is generated by applying D_{120} to the first cube to create the second and again to produce the third orientation. Applying X_{180} to the first cube produces the fourth cube in the top row. Applying D_{120} to the fourth cube produces the fifth cube and applying D_{120} again generates the sixth cube in the top row. Applying R_{90} to any cube produces the cube immediately below it.

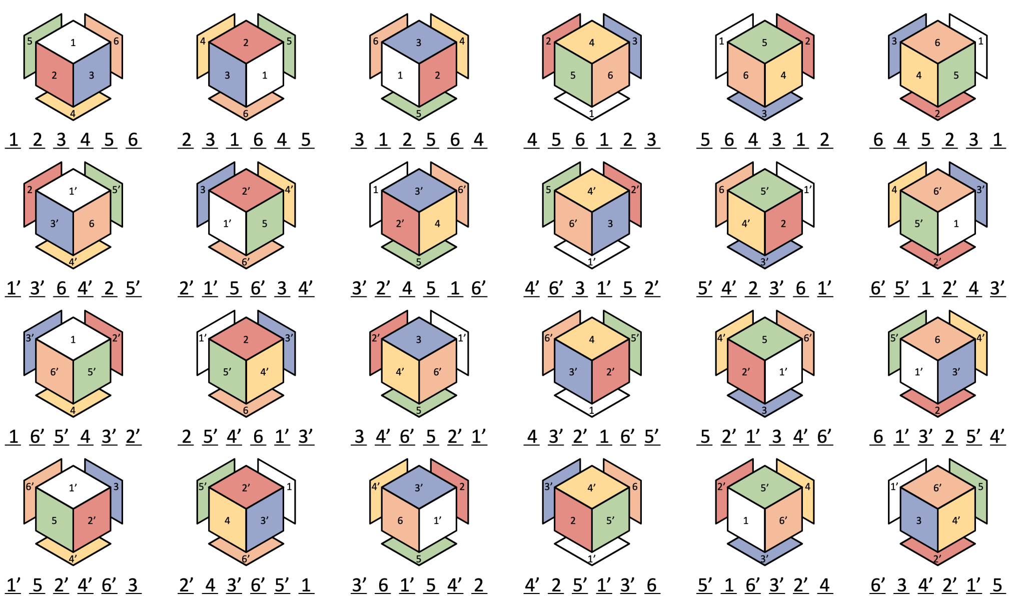


Figure 9: *The result of all 24 rotations of the cube on a generic labeling.*

Re-Framing the Problem and Results

Based on our cordon labeling system and our insight into how rotations of the cube affect the labelings, we can re-frame our research question in terms of the group of rotations of the cube acting on set of all 6-letter words using the alphabet {A, B, C, D}. We can consider the collection of labelings generated from rotating a cordon to all possible orientations. This set of labelings is called the ***orbit*** of that cordon under the group action. For example, Figure 6 shows the orbit of the cordon that can be labeled with ABCDDD when acted on by the group of rotations of a cube. This shows us that the same cordon could also be labeled with any of those 24 labelings. Figure 9 shows the orbit of a generic element in S under the group action. Under group actions, orbits form equivalence classes and are either equal or disjoint. So, if two cordons can be rotated to have the same labeling, then their orbits under the group action are equal. Our research question can then be re-phrased to, "*How many distinct orbits of this group action exist?*"

Although we are aware of more advanced and/or elegant methods [1], in this project we computed the orbit of all 4,096 labelings using Microsoft Excel and identified which orbits were repetitions of previously listed orbits. We first identified the "least" labeling in each orbit – the first labeling if all labelings in that orbit were listed alphabetically. Because orbits are either equal or disjoint, if two orbits share any labeling, they have the same least labeling. We identified 224 distinct least labelings (Table 1). No two of these labelings are contained in the same orbit, meaning there are 224 distinct orbits under this group action. More pointedly, all 4,096 cordons (and, so, patterns) can be reoriented and labeled so that their labeling is listed in Table 1.

Table 1: *Alphabetized list of 224 least labelings of distinct orbits under the group action.*

AAAAAA	AABABC	AABCDC	AACBDA	AACDDD	AADDCD	ACBCAD	ACDADC
AAAAAB	AABABD	AABCDD	AACBDB	AADAAD	AADDDA	ACBCCC	ACDBCC
AAAAAC	AABACC	AABDAB	AACBDC	AADACB	AADDDC	ACBCCD	ACDBCD
AAAAAD	AABACD	AABDAC	AACBDD	AADACC	AADDDD	ACBCDC	ACDCCC
AAAABB	AABADC	AABDAD	AACCAA	AADACD	ABCBCA	ACBCDD	ACDCCD
AAAABC	AABBAA	AABDBC	AACCAC	AADADA	ABCBCC	ACBDAD	ACDCDC
AAAABD	AABBAB	AABDBD	AACCAD	AADADC	ABCBCD	ACBDCC	ACDDAC
AAAACB	AABBAC	AABDCA	AACCBC	AADADD	ABCBDA	ACBDCD	ACDDCC
AAAACC	AABBAD	AABDCB	AACCBD	AADBCA	ABCBDC	ACBDDD	ACDDCD
AAAACD	AABBCA	AABDCC	AACCCA	AADBCB	ABCBDD	ACCACC	ACDDDC
AAAADB	AABBCB	AABDCD	AACCCB	AADBCC	ABCCCC	ACCACD	ADCADC
AAAADC	AABBCC	AABDDB	AACCCC	AADBCD	ABCCCD	ACCADC	ADCBCC
AAAADD	AABBCD	AABDDC	AACCCD	AADCAC	ABCCDA	ACCADD	ADCBDC
AAABBB	AABBDA	AABDDD	AACCDA	AADCAD	ABCCDC	ACCBCC	ADCCCC
AAABBC	AABBDB	AACAAC	AACCDB	AADCBC	ABCCDD	ACCBCD	ADCCDC
AAABBD	AABBDC	AACAAD	AACCDC	AADCCA	ABCDCC	ACCBDC	ADCDCC
AAABCC	AABBDD	AACACA	AACCDD	AADCCB	ABCDDA	ACCBDD	ADCDDC
AAABCD	AABCAB	AACACB	AACDAA	AADCCC	ABCDDC	ACCCAC	ADDBCC
AAABDC	AABCAC	AACACC	AACDAC	AADCCD	ABCDDD	ACCCCC	ADDCCC
AAABDD	AABCAD	AACACD	AACDAD	AADCDA	ABDCDC	ACCCCD	ADDDCC
AAACCC	AABCBC	AACADA	AACDBD	AADCDC	ABDCDD	ACCCDC	CCCCCC
AAACCD	AABCBD	AACADB	AACDCA	AADCDD	ABDDDC	ACCCDD	CCCCCD
AAACDD	AABCCA	AACADC	AACDCB	AADDAA	ACBBAC	ACCDAC	CCCCDD
AAADDD	AABCCB	AACADD	AACDCC	AADDAC	ACBBAD	ACCDCC	CCCDDD
AABAAB	AABCCC	AACBCA	AACDCD	AADDAD	ACBBCC	ACCDCD	CCDCCD
AABAAC	AABCCD	AACBCB	AACDDA	AADDCA	ACBBCD	ACCDDC	CCDCDC
AABAAD	AABCDA	AACBCC	AACDDB	AADDCB	ACBBDC	ACCDDD	CCDDCC
AABABA	AABCDB	AACBCD	AACDDC	AADDCC	ACBBDD	ACDACD	CCDDCD

We have colored the entries in Table 1 to indicate the number of distinct labelings in each orbit (red indicates orbits of size 2, orange: 4, yellow: 6, green: 8, blue: 12, and white: 24). We noticed that exactly 2 orbits contain 2 distinct labelings, 4 orbits contain 4 distinct labelings, 18 orbits contain 6 distinct labelings, 4 orbits contain 8 distinct labelings, 64 orbits contain 12 distinct labelings, and 132 orbits contain 24 distinct labelings. We will now use two orbits to illustrate how the symmetries of the cube interact with the labelings to produce orbits of different sizes: the orbit with least labeling AAABBB (which contains 2 distinct labelings) and the orbit with least labeling CCDCCD (which contains 6 distinct labelings).

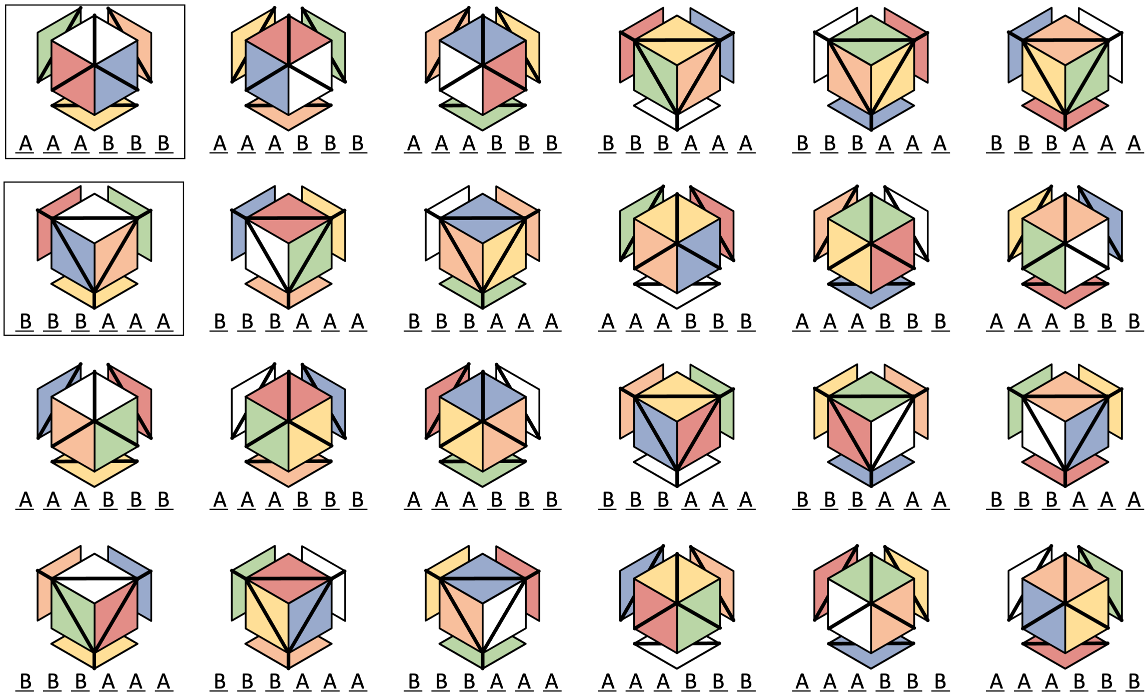


Figure 10: *The orbit of AAABBB contains exactly two distinct labelings.*

Early on in our work, we recognized that some of the knot patterns had several symmetries. This included the pattern that we now label as having the cordon AAABBB. Figure 10 shows all 24 symmetries of the cube with this cordon. Notice that there are exactly 2 distinct labelings throughout the entire orbit. This is due to the structure of the cordon, which forms a tetrahedron whose vertices are on alternating corners of the cube. Because of this, the cordon is incident with one half of the cube's corners (as with AAABBB) or the other half of the cube's corners (as with BBBAAA). We were drawn to this example because the geometric structure gives insight into how the group action creates an orbit with only two distinct labelings. The symmetries of the tetrahedron interact with the symmetries of the cube so that there are only two possible cordons. Interestingly, the knot pattern associated with this cordon is four disjoint trefoil knots (Figure 11).

Figure 11: *Pattern with associated labeling AAABBB from one perspective showing a Trefoil knot.*

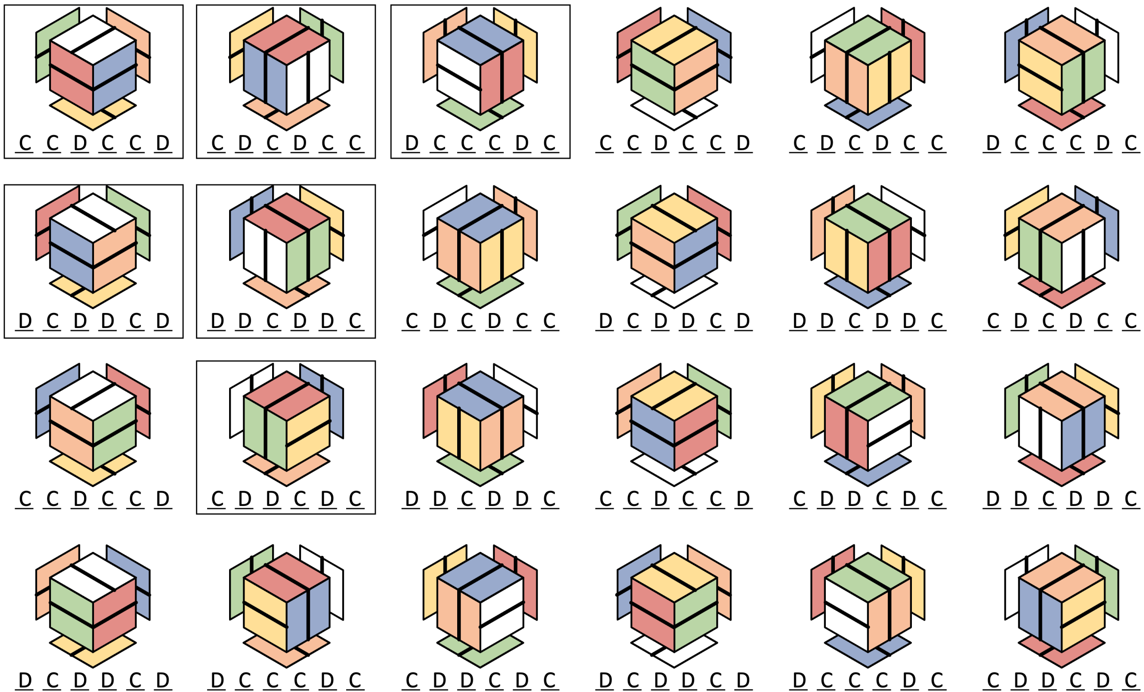


Figure 12: *The orbit of CCDCCD contains exactly six distinct labelings.*

The final example we identified as particularly enlightening is the cordon with least labeling CCDCCD. This orbit contains six distinct labelings (highlighted with rectangles in Figure 12). This cordon is made up of a connected band around four sides and a single line on the remaining two sides, each of which always has the same label. The symmetries of this cordon are dictated by the axis about which the band is connected (x, y, or z) and the label of the other two lines (C or D). This yields $3\times2 = 6$ possible orientations of the cordon. As before, attending to the geometric structure gives insight into the number of distinct labelings in this orbit. The band separates the cube into two halves, causing the resulting knot structure to be two disjoint links, each with 6 crossings. In our future work, we will analyze the knots that the 224 corresponding patterns generate and identify how many distinct knots make up this class of photogenic knots on the $9\times9\times9$ Rubik's cube.

Acknowledgements

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References

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