Interlace Patterns Emerging in a Penrose-Type Islamic Design

Joseph I. Cline

Department of Chemistry, University of Nevada, Reno, USA; cline@unr.edu

Abstract

An aperiodic Penrose P2 tiling is decorated following the traditional Girih rules of medieval Islamic geometric art. The resulting strapwork strands form an apparently infinite number of closed loop shapes, simple knots, and complex knots. Organizing principles are identified that describe symmetries and patterns of self-similarity observed among selected strands. These characteristics are quite different from the small number of distinct strands observed in traditionally decorated Islamic periodic tilings.

Introduction

Medieval Islamic art and architecture is renowned for its beautifully patterned tilings. Although there are many notable exceptions, most commonly these patterns decorate *periodic* tilings of the plane. The decorations are often polygons defined by interlaced strapwork which runs through the composition. The polygon arrangements have appealing symmetries, and the intricate, interlaced strapwork both bewilders and delights the eye. One can choose a particular strapwork path or "strand" and marvel at the complexity of its trajectory as it weaves through the composition.

This paper describes strands in decorative strapwork that obey a particular "golden rule" often observed in Girih decoration: the strapwork never bends at crossing intersections, and there are no terminations or "T" intersections [3][4]. When realized in three dimensions, the strapwork is interlaced or woven such that in following a strand there are alternate "over" and "under" strapwork crossings along its trajectory. For *periodic* tilings, Grünbaum and Shephard devised a general method using Cayley diagrams to analyze such Girih strapwork strands [8]. They showed that, for periodic tilings, there are a small number of unique strand shapes, which they call interlace patterns, arranged periodically throughout the composition. The interlace patterns form either finite, closed loops or infinitely long, unbounded strands consisting of concatenated repeat units that run across the composition. Their method also characterizes interlace patterns by the number of over-under crossings encountered around a looped strand, or the number of such crossings in the repeat unit of an unbounded strand. They report a survey of Islamic Girih strapwork showing that most medieval compositions contain fewer than 4 strand shapes. Grünbaum and Shephard give worked-out examples of their method to analyze tilings of symmetry groups $p4m$ and $p6m$. These examples have crossing numbers up to 138 for closed loop strands and up to 120 for the repeat units of unbounded strands. Ostromoukov provided details for extending the Cayley diagram method to analyze compositions based on all 17 planar symmetry (wallpaper) groups [10].

This paper provides a sampling from a survey of the interlace patterns observed when a simple Girih decoration rule is applied to an *aperiodic* (Penrose P2) tiling. In contrast to periodic tilings, it appears that all strands are closed loops, with no obvious limits to the number of unique strand shapes and crossing numbers. While some strands show no self-crossings, many show a very large number of self-crossings. If the interlacing pattern is interpreted in three dimensions, these self-crossing strands can form highly complex mathematical knots.

Girih Decoration Rule

The kite and dart shapes comprising the well-known Penrose P2 tiling, defined elsewhere [5][6][7], are shown in figure 1(a,b). Here we define a to be the length of the long sides of the tiles, so that the short sides have length $\frac{a}{\phi}$, where ϕ is the Golden Ratio, $(1+\sqrt{5})/2 \approx 1.6180$.

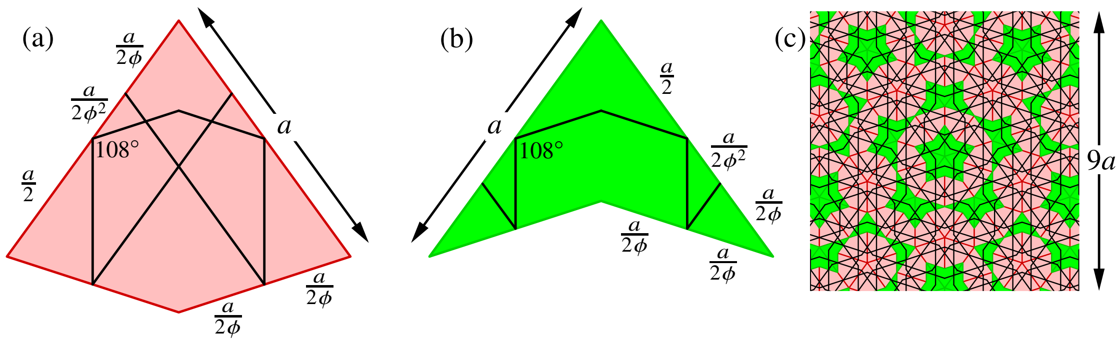


Figure 1: *Girih decoration rule for the P2 tiling. (a) Kite with Girih decoration shown as black lines. (b) Decorated dart. (c) A decorated region of the tiling generated by inflation of the "star" vertex neighborhood [7]. The resulting tiling has d5 symmetry about the 5-pointed star at its center.*

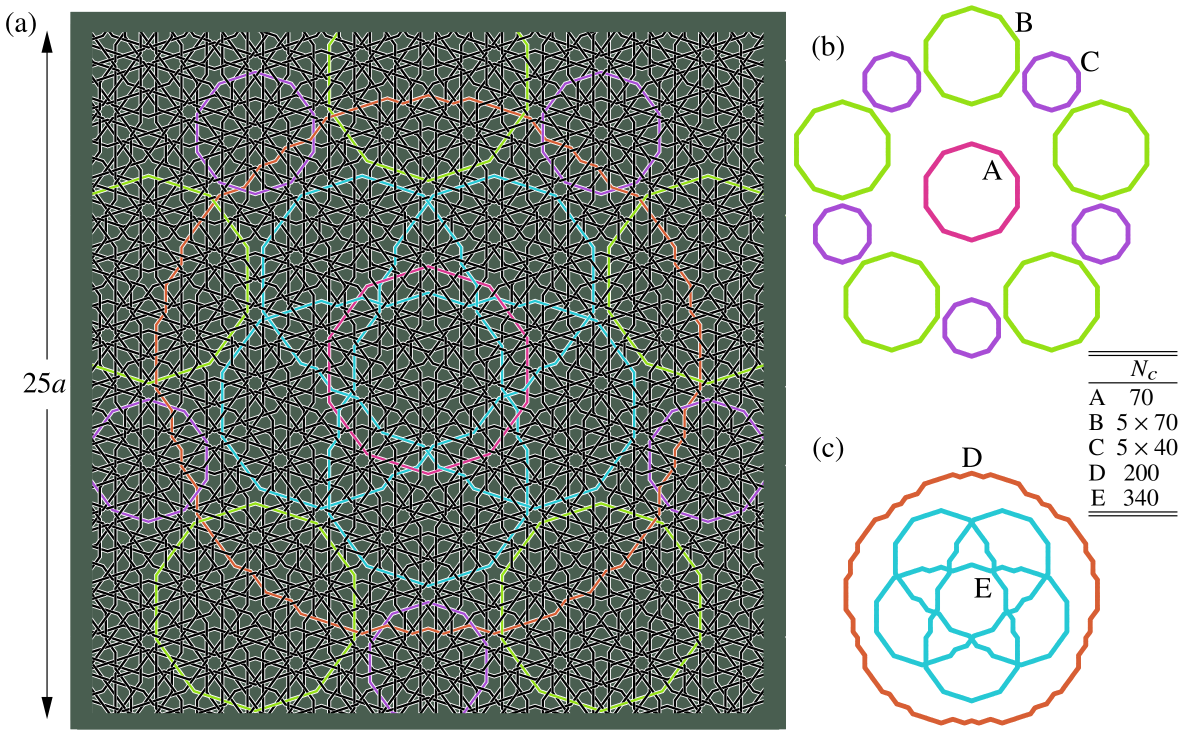
There are several reports of Penrose-type Islamic patterns generated by Girih-decorated kites and darts assembled into a Penrose P2 tiling [9][11][12][13]. In the work most closely related to this paper, Rigby and Wichmann chose decorations designed to generate traditional Girih motifs (*e.g.* stars and complete rosettes) in the resulting compositions [12][13]. However, the patches they report are too small to explore the shapes of the strands that emerge in the resulting strapwork. The Girih decoration rule used in this paper is much simpler than those in these previous reports, with some sacrifice of fidelity to traditional design. Figure 1(a,b) shows our decoration rule as black lines superimposed onto the kites and darts.

This work only considers decoration of the Penrose tiling generated by an even number of inflations of the "star" vertex neighborhood (see, for example, Fig. 10.3.20 of [7], page 12 of [6], or the "infinite star pattern" of [5]). Figure 1(c) shows the central region of the resulting "star axiom" tiling, which, though aperiodic, has $d5$ symmetry about the five-pointed star formed by the 5 darts sharing a vertex at its center. The black lines in figure 1(c) show the skeletal form of the resulting strapwork pattern, omitting the depiction of interlacing at strand crossings.

A more expansive view of this region is shown in figure 2(a), which omits the underlying P2 tessellation to focus on the strapwork pattern. Enlargement and close inspection of figure 2(a) shows the alternate over-under interlacing of the strapwork strands. Interlacing destroys the $d5$ reflection planes, but retains the 5-fold rotational symmetry about the center star axiom [1].

Strand Analysis

Only a few strands can be completely followed in figure 2(a); most run off its edges. Five strand shapes are highlighted by unique colors. For clarity these strands are extracted from the strapwork and reproduced in figure 2(b,c). These strands exhibit properties common to all strands observed in the survey of this pattern: *All strand shapes are closed loops. These loops have at least 5-fold rotational symmetry about the center of the composition, or are members of a 5-fold rotationally symmetric irreducible representation of the composition's point group. Ignoring interlacing, each strand or 5-fold degenerate representation has 5 reflection planes through the center of the star axiom tiling.*



	N_c
A	70
B	5×70
C	5×40
D	200
E	340

Figure 2: *Decoration in the central region of the star axiom tiling. (a) Interlaced strapwork with selected strands highlighted in color. (b,c) Shapes of 5 simple strands with small N_c values, labeled A-E.*

These colored strands, labeled A-E in figure 2(b,c), are among the simplest strand shapes found in the composition. Strand A is a regular decagon. Defining the crossing number, N_c, to be the number of crossings encountered when completing the circuit of a single strand's loop, we have $N_c = 70$ for strand A. Strands B are part of a 5-fold degenerate representation of regular decagons, each identical to strand A (and also with $N_c = 70$). Close inspection of figure 2(a) shows that the strand B decagons enclose regions with strapwork identical to that enclosed by strand A (after rotation of the enclosed region by 36°). This observation is of general utility: strand shapes allow rapid identification of local symmetry centers in the composition.

Strand C has $N_c = 40$, the smallest of any strand shape in the composition. It is also comprised of a 5-fold degenerate representation of decagons. Strand D has $N_c = 200$ and forms a loop around the composition's center with 5-fold rotational symmetry. Strand E has $N_c = 340$ and has 10 self-crossings. Close inspection of its self-crossings in figure 2(a) shows that strand E forms a $T(5,3)$ torus knot.

Figure 2(a) depicts a roughly $25a \times 25a$ region. Strands generally traverse a much larger region, and a large tiling is required to begin exploring the variety of their shapes. The tiling analyzed in this work was generated by inflating the star axiom 18 times [2], giving a usable tiling radius of $a\phi^{17} \approx 3571a$. Within this radius the tiling contains approximately 30 million darts and 49 million kites, with roughly 300 million strand crossings in the strapwork pattern. No effort was made to estimate the number of individual strands within this radius. Only a minuscule fraction of these strands were examined, mostly in the central region of the tiling. The trajectories of the selected strands were computationally followed [2]. Of those examined in this survey, only a representative sampling can be reported here.

One might begin a strand taxonomy by defining two broad categories: those "captured in orbit" around local centers of symmetry, and those delocalized and "orbiting" the tiling's center. Strands in the latter category can explore large areas of the tiling, winding through and partitioning domains of local symmetry, and can have N_c values much larger than those of strands A-E in figure 2. Examination also shows that local centers of symmetry and extrema of strand trajectories are frequently located on rings of radius $a\phi^n$ centered on the central star axiom, where $n = 0, 1, 2, \ldots$

We now consider strands in this second, delocalized category. Many can be grouped into families of strands having similar shapes, but with sizes scaled according to the $r = a\phi^n$ organizational principle. As an example, strand E, the torus knot in figure 2(c), is the smallest member of such a family. Its first 6 members are shown in figure 3, with visualization of their relative scaling aided by concentric "index rings" of radius $a\phi^n$. In figure 3, the inner trajectory of each family member is roughly tangent to an "inner index ring" with $n_i = 2, 4, 6 \ldots$ ("Inner index" n_i values provide convenient labels to distinguish individual members of strand families.) All have the same general shape, but with each successive member rotated by 36° from its predecessor. For larger members, the small scale zig-zag corrugations of the strand trajectory become insignificant relative to the overall scale of its shape, with straight segments of the strand trajectory arising from Conway worms [7] in the underlying Penrose tiling. The N_c values of successive members grow nearly exponentially by factors of ϕ^2. For example, the two largest family members in figure 3 have an N_c ratio of 45,550/17,380, within 0.1% of ϕ^2. The strand survey showed no apparent limit to the number of members of this family, and the N_c ratio for successive members appears to approach ϕ^2 exactly. This behavior was typical of most delocalized strand families observed.

inner index ring, n_i	N_c
2	340
4	940
6	2,510
8	6,620
10	17,380
12	45,550

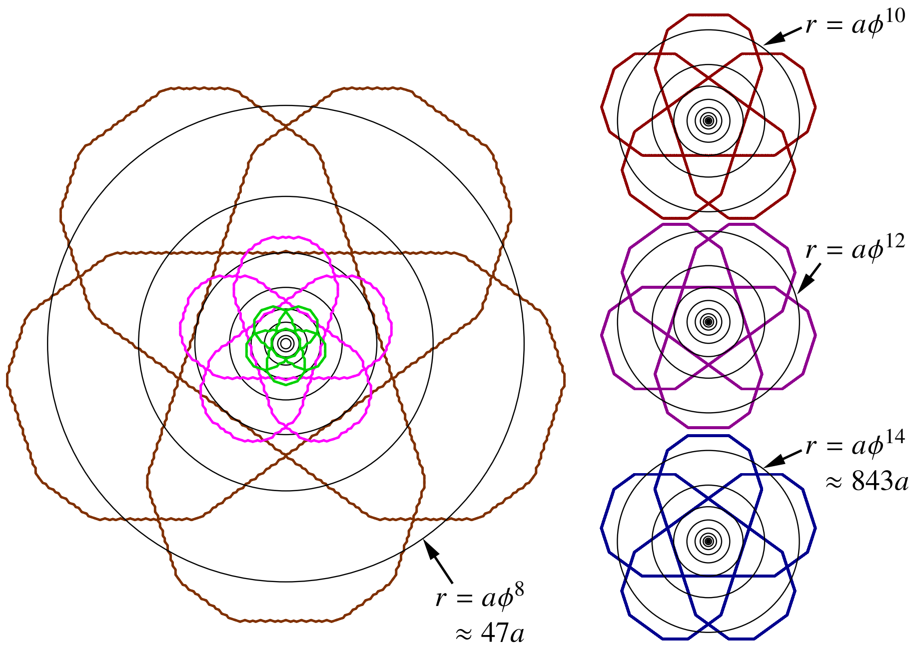
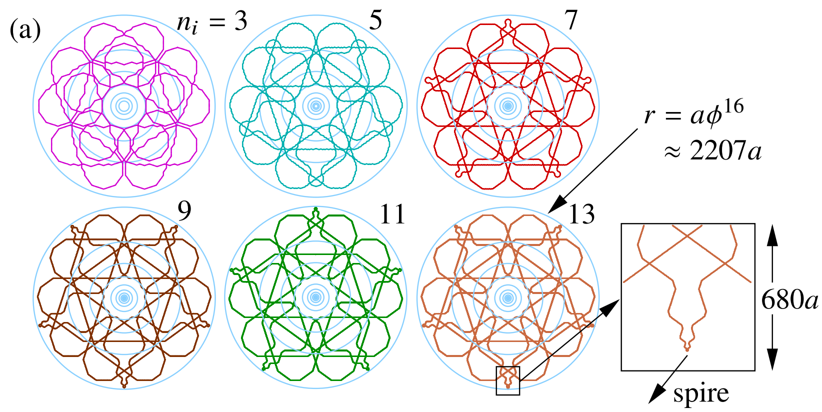


Figure 3: *The shapes of the 6 smallest members of the strand family that begins with strand E in figure 2(c), shown with $a\phi^n$ index rings and a tabulation of their crossing numbers, N_c.*

The sizes of successive members of the knotted strand family shown in figure 4(a) also grow exponentially by ϕ^2, however, unlike the family in figure 3, they show an evolution in shape. Each member has an inner trajectory roughly tangent to an inner index ring with $n_i = 3, 5, 7 \ldots$, and is rotated by 36° from its predecessor. A radially directed "tower" feature develops in each 72° sector as n_i increases, with the outer "spire" of the tower nearly touching the index ring with $n = n_i + 3$. While the size and shape of these spires are constant, each successive family member has one additional "floor" below the spires of its five towers, with the width of each floor very close to a factor of ϕ^2 wider than the one above it. For larger values of n_i, the overall size of the strand dwarfs the spires of its towers, concealing the fine structure of its spires in figure 4.

inner index ring, n_i	N_c
3	1530
5	4470
7	12,230
9	32,610
11	86,030
13	225,950



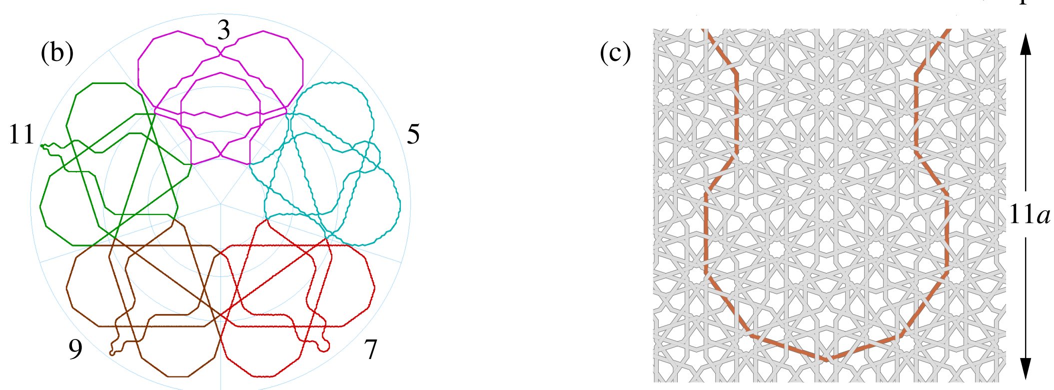


Figure 4: *(a) The 6 smallest members of a strand family, each labeled by the exponent of its inner index ring, n_i. (b) Sector diagram showing tower development. (c) Tower spire magnification for $n_i = 13$.*

Figure 4(c) magnifies one spire of the family member with $n_i = 13$, and close inspection shows the fractal-like development of the tower floors and the detailed shape of the strand trajectory in the spire region as it weaves through the interlaced Islamic pattern. Unsurprisingly, successive family members have N_c values that increase nearly exponentially by a factor of ϕ^2. All members of the family in figure 4 are self-crossing strands forming torus-like knots orbiting the star axiom at the center of the tiling; the characterization of these knots is beyond the scope of this paper.

The "sector diagram" in figure 4(b) provides a compact representation of the evolution of the first five members of this family, facilitating a direct comparison of their shapes and scaling. Moving clockwise in the diagram from the first family member's sector (with $n_i = 3$), each 72° sector shows the successive member with its size scaled by a successive factor of $1/\phi^2$.

Sector diagrams of this type are shown in figure 5(a,b) for two related strand families. In figure 5(a,b) individual family members are labeled by their inner index ring values, n_i, and tabulation of their N_c values again shows exponential successive increases by factors approaching ϕ^2. Figure 5(a) shows the sector diagram for a strand family forming a torus-like knot surrounding the center of the tiling. This family has *two* evolving towers in each 72° sector, with a "pretzel" centered on its inner index ring. Figure 5(b) shows the sector diagram for a strand family with each member part of 5-fold symmetric representation of the $d5$ point group. Each 72° sector of these strands contains *four* evolving towers.

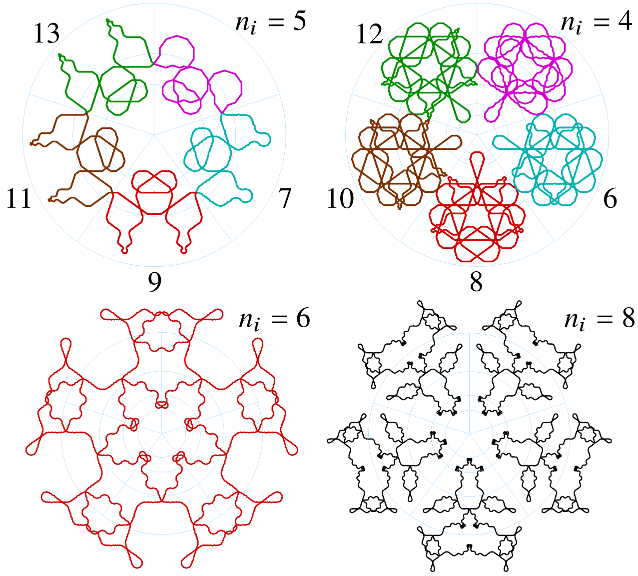
Pretzels are especially common features among strand families, however the "pretzels" and "towers" observed in figures 4 and 5(a,b) are only two examples of many structural themes observed in the survey of

(a)

n_i	N_c
5	1500
7	4700
9	13,210
11	35,620
13	94,420

(b)

n_i	N_c
4	5×1344
6	5×3906
8	5×10,664
10	5×28,408
12	5×74,914



(c)

n_i	N_c
6	7510
8	20,770
10	55,810
12	147,870

(d)

n_i	N_c
8	5×5608
10	5×15,560
12	5×51,876

Figure 5: *(a) Sector diagram for a family with pretzels and 10 towers. (b) Sector diagram for a family of 5-fold representations of* $d5$ *with each member containing* $5 \times 4 = 20$ *towers. (c) Smallest member* ($n_i = 6$) *of a family with pretzels and bears. (d) Smallest member* ($n_i = 8$) *of a 5-fold family with pretzels, bears, and cubs.*

strand shapes arising in the strapwork of this Islamic pattern. Figure 5(c,d) shows the smallest members of two interesting strand families. We recognize pretzels, along with a new structural theme resembling a "bear" with two ears. The family in figure 5(d) also has smaller, one-eared "cubs". In the survey of strand shapes, the appearance of cub features is observed to be associated with the presence of bears in large strands.

As n_i increases, the N_c values of the four families in figure 5 become very large, and it was not possible to analyze larger members of these families due to the finite size of the star axiom tiling generated by 18 iterations of inflation. Figure 6 shows the most intricate *complete* strand observed in the survey that did not run off the edge of this tiling. It is a 5-fold degenerate representation of $d5$ with each component strand tangent to an inner index ring $n_i = 12$ and extending beyond the $n = 15$ index ring. Each of the five strands has $N_c = 452{,}618$ for a total of $5 \times N_c = 2{,}263{,}090$ crossings in the representation. Larger N_c values were seen for single-strand, nondegenerate representations in the survey, but the combined number of crossings in figure 6 was the largest observed for one irreducible representation.

The strands in figure 6 contain a plethora of pretzels, bears, cubs, and many other features either too small to be resolved on the scale of the figure, or lost in the complexity of the many self-crossings, even after the magnification shown in figure 6. Some intricate features with roughly circular boundaries are the result of near-capture of the strand trajectory in complex "orbits" around local centers of symmetry in the underlying Penrose tiling. Close inspection of the strand trajectory in the vicinity of these local orbits implies the existence of "partner" strands which would complete a shell around a local symmetry center. The survey indeed revealed many instances of such sets of partner strands around local symmetry centers, however providing examples of these structures is beyond the scope of the present paper.

Summary and Conclusions

This survey of strand shapes is restricted to a single decoration rule applied to one particular Penrose tiling with a 5-fold center of symmetry. The tiling symmetry necessarily constrains the strands of the resulting

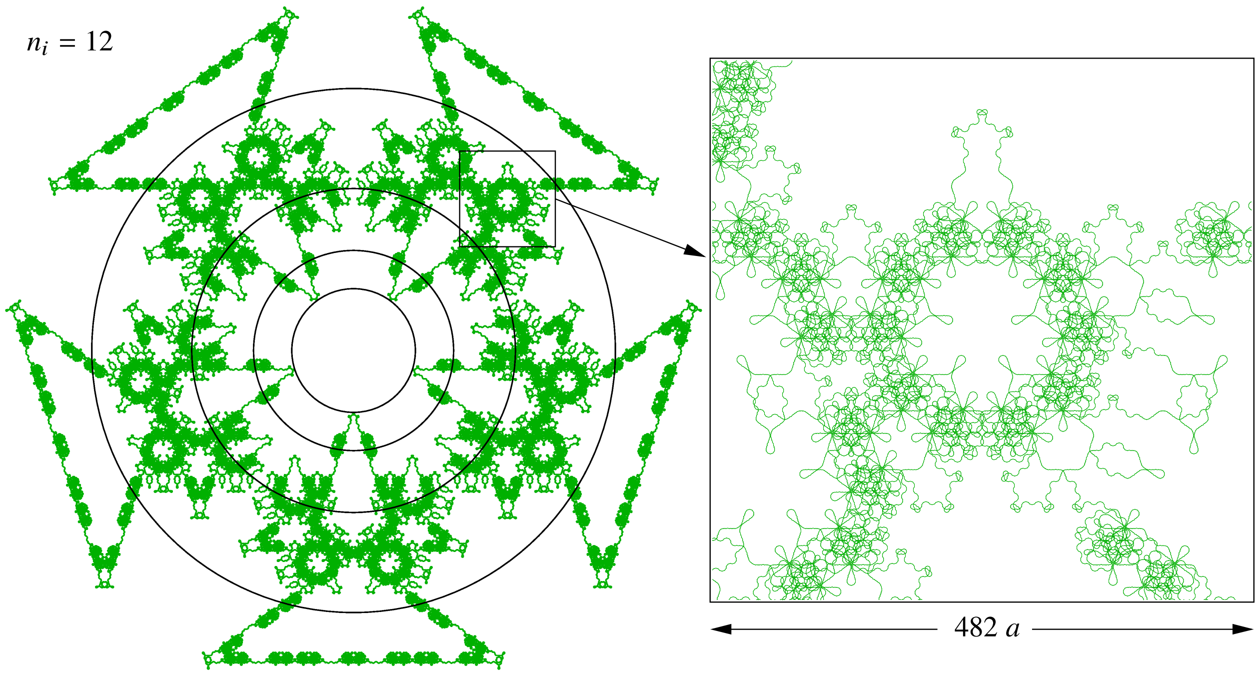


Figure 6: *A 5-fold degenerate set of strands forming a representation of the d5 point group about the center of the tiling. Each strand has N_c = 452,618 crossings for a total of $5 \times N_c$ = 2,263,090 crossings.*

strapwork pattern to be representations of the *d*5 point group, having a 5-fold center of rotation (and 5 reflection planes ignoring interlacing). All strands observed in the survey either form closed loops, or, if they ran off the edge of the tiling, appeared to be large members of a family of similar strands consisting of closed loops, suggesting the strand would form a loop on a sufficiently large tiling. Many form knots with complex shapes and many self-crossings. I conjecture that there is no limit on the variety of strand shapes or strand crossing numbers. The relatively few strands and features highlighted in this paper provide a glimpse of the rich and aesthetically pleasing set of shapes and structural themes encountered in the survey. Girih strapwork strands decorating other Penrose P2 tilings also will be expected to form closed loops, with the single exception of the Penrose cartwheel tiling. It contains 10 semi-infinite Conway worms [7]. The strands tracking these 10 special Conway worms of the cartwheel tiling could not form loops.

The decoration rule chosen here is relatively simple compared to those others have used for Penrose tilings [9][11][12][13]. It would be interesting to discover the resulting strapwork strand shapes in those patterns, which have more traditional Girih motifs. Provided that such patterns meet the constraint of linear crossings with no terminations or T-intersections, it seems likely they also will form simple loops and complex knots with large crossing numbers, with no obvious limit.

For Girih decorations of *periodic* tilings, Grünbaum and Shephard were able to provide a general, analytic method to completely describe the strapwork strands, without a need to assemble the pattern [8]. Their survey of traditional tilings shows a small (typically 4 or fewer), finite number of strand shapes and relatively small crossing numbers. It is not obvious to this author that an analogous general analytic method exists for Girih decorations on *aperiodic* tilings, given that there appears to be no limit on the number of possible shapes. The complexity and unpredictability of Girih strapwork patterns on aperiodic tilings seems an emergent feature arising from the interaction of the decoration rule and the underlying tiling - providing challenges to the interested mathematician and delights for admirers of the aesthetics of Islamic patterns.

The results of this survey offer a few empirical guidelines of predictive value for strands emerging in Girih decoration of Penrose tilings. Most importantly, it is frequently observed that characteristic structural features of strands are consistently located at radial distances $r = a\phi^n$, $n = 0, 1, 2 \ldots$ from centers of symmetry of the tiling. This behavior facilitates identification of the members of strand families with similar shapes that would otherwise be lost in the vast strapwork network. It is observed that successive members of such families have sizes and crossing numbers that increase by a factor of ϕ^2, with each new member rotated by 36° from the center of symmetry of the family. Many strand families identified in the survey simply grow successively in size by ϕ^2. Other families show shape evolution with features having fractal characteristics.

Penrose tilings delight the eye with a mix of local order and long range chaos. The strand shapes found in Girih interlace patterns on such tilings offer the visualization of large, delocalized structures amid this chaos. Strand trajectories readily identify local centers of symmetry, the relative locations of these centers, and the extent of their self-similarity. When extracted from the interlace pattern, the knots formed by these strands have their own aesthetic and mathematical appeal and are worthy of further investigation.

References

[1] The over/under configuration at one strapwork crossing constrains the configuration at all other crossings. Interaced strapwork is chiral; mirror plane reflection exchanges all crossings.

[2] Perl scripts were written to inflate and decorate Penrose tilings, and to follow strands and count crossings. The $d5$ symmetry of the skeletal strapwork reduced computational effort. Tiling patterns were stored as `fig` code (https://mcj.sourceforge.net/fig-format.html) and graphically rendered by `fig2dev`.

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