Large Islamic Rosettes in an Octagonal Frame

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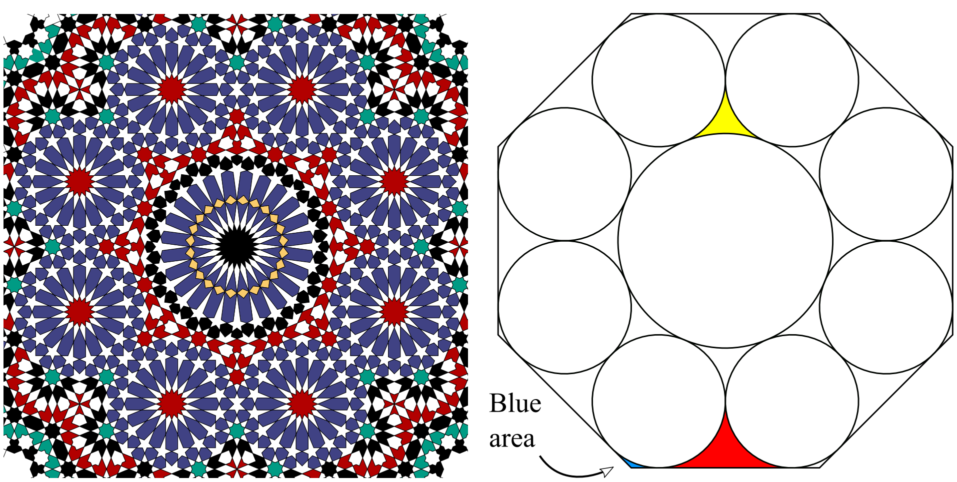
Abstract

Many traditional Islamic geometric patterns feature a single large rosette surrounded by a ring of eight satellite rosettes. We present a technique for constructing patterns in this style, which scales to rosettes larger than those executed historically. We decompose the regions between these rosettes into different characteristic regions called basic shapes, and show how to fill each basic shape with polygons that are close to regular, suitable for motif construction using the polygons-in-contact method.

Introduction

Islamic geometric patterns are usually built around arrangements of rotationally symmetric motifs like stars and rosettes. A common design goal is to create a composition centered on a single large rosette. For example, the design in Figure 1(a) is built around a central 24-armed rosette; the rosette has a black star at its core, surrounded by 24 blue "petals", each ornamented with a small yellow kite. Eight 16-armed rosettes surround the center.

Historical patterns include rosettes with as many as 96 points; Castera [4] offers a large number of examples of rosettes of many sizes. He also demonstrates the challenges involved in reconciling these highly symmetric stars and rosettes with a surrounding pattern, which usually consists of much simpler shapes derived from an eight-pointed star.



(a) **(b)**

Figure 1: *(a) An Islamic pattern with a central 24-armed rosette and eight 16-armed satellite rosettes, adapted from a drawing by Brian Wichmann [8]; (b) a schematic diagram of this type of composition.*

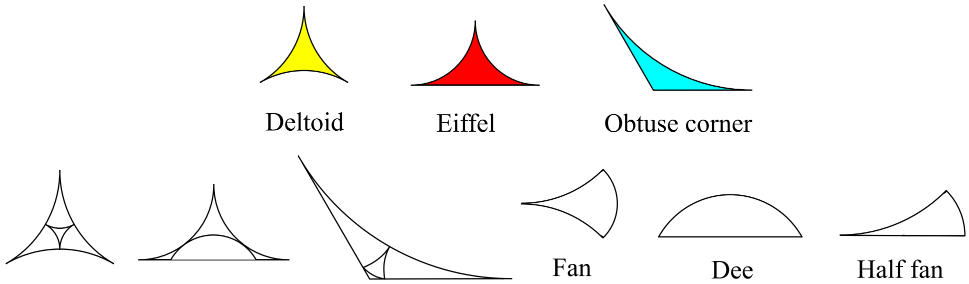


Figure 2: *Basic shapes.*

In our work we seek to venture far beyond 96, creating new patterns that feature central rosettes with hundreds or thousands of points. Larger rosettes tend to worsen the distortion of individual motifs in the transition zone to the surrounding pattern, creating new opportunities for research that distributes, hides, or minimizes that distortion.

Previously, we avoided many of these issues by surrounding a large rosette with concentric rings of new geometry, yielding results that are aesthetically interesting but unconventional [1]. Here we adhere more closely to the structure of traditional patterns. In particular, we observe that a large central rosette is almost always surrounded by a ring of eight "satellite" rosettes of an intermediate size. The example in Figure 1(a) has this form, which is also illustrated schematically in Figure 1(b). It follows naturally that the central rosette's size should be a multiple of 8. Consider the current year, for example: because $2024 = 8 \times 253$, we might wish to create a design with a central 2024-pointed rosette. We have developed a construction technique tailored to layouts like these. It is a computer-assisted manual technique, based on decomposing the interstitial regions between the central rosette and its satellites into manageable "basic shapes" (shown in Figure 1(b)) and developing standardized strategies for filling those shapes with close-to-regular polygons. The polygons can then be decorated with traditional motifs using the polygons-in-contact method [3, 6]. In this paper we present our technique and apply it to the example of a 192-armed rosette with 112-armed satellite rosettes, all contained within an octagonal frame.

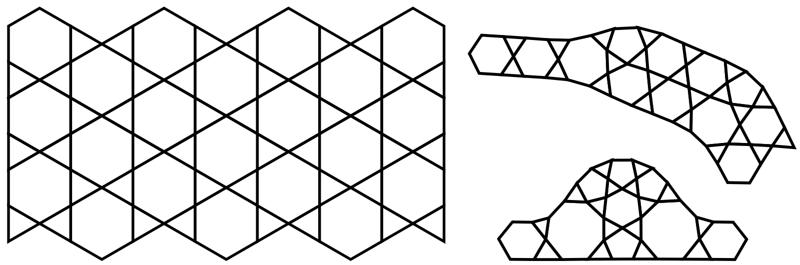
Pattern Anatomy

In the polygons-in-contact method, a rosette is applied as a motif inside a regular polygon. As the size of the rosette increases, its polygon more closely approximates a circle. We therefore begin the design process with the abstract schematic of a circle surrounded by a ring of eight smaller tangent circles. We place the circles inside an octagonal frame, which will ease the transition to any surrounding motifs. This schematic is visualized in Figure 1(b).

This starting arrangement leaves three distinct shapes of interstitial regions to be filled with polygons. One example of each shape is coloured: the *deltoid* (yellow), the *eiffel* (red), and the *obtuse corner* (blue). They are also shown in isolation in the top row of Figure 2. Each is bounded by three curves that are either circular arcs or straight lines. We further decompose these shapes as shown in the bottom left of Figure 2, resulting in three additional "basic shapes" shown on the bottom right: the *fan*, the *dee*, and the *half fan*. We will develop specialized strategies for filling these basic shapes with near-regular polygons. Note that these shapes are an idealized visualization. Although the fan and half fan have cusps, representing tangencies between neighboring circles, our patterns will include a small channel of polygons running between the large regular polygons associated with the rosettes.

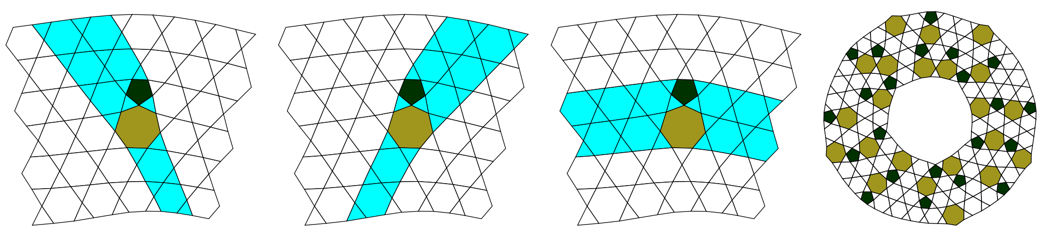
Modified [3.6.3.6] Patterns

In this work we fill basic shapes using modifications of the Archimedean tiling [3.6.3.6] illustrated in Figure 3(a), in which every vertex is surrounded by alternating equilateral triangles and regular hexagons. This tiling features fault lines in three directions, which allows it to adapt naturally to straight or gently curved boundaries in the structure of a pattern. We divert these fault lines by introducing defects in the tiling. Specifically, we replace hexagons by polygons with five or more sides, and triangles by squares, as illustrated in the examples in Figure 3(b).



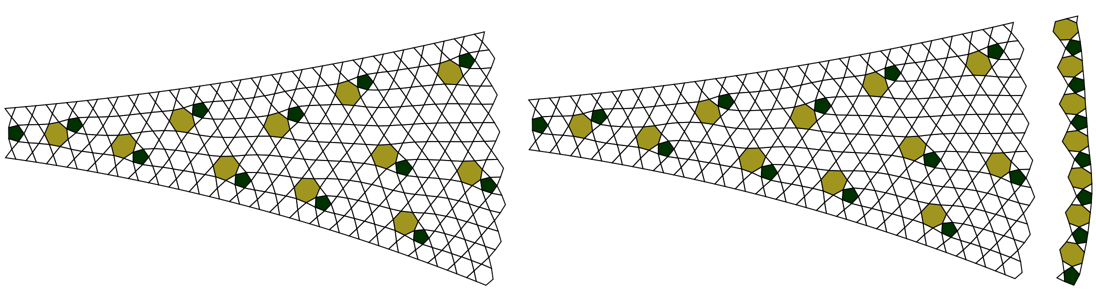
(a) **(b)**

Figure 3: *(a) A* [3.6.3.6] *tiling; (b) some modified patches of tiles that arise when individual tiles are replaced by other near-regular polygons.*



(a) **(b)** **(c)** **(d)**

Figure 4: *Modifying the* [3.6.3.6] *tiling with 7-5 pairs. (a,b) A 7-5 pair creates a branch in two fault line directions, (c) which lengthens rows of hexagons in the third direction. (d) A series of 7-5 pairs can bend a patch of tiles into an annulus.*



(a) **(b)**

Figure 5: *(a) A wedge-shaped patch of modified [3.6.3.6], with a cap (b) to create a smooth end.*

These modified [3.6.3.6] tilings are highly adaptable. For example, Figure 4 shows how the surrounding tiling behaves when two neighboring hexagons are replaced by a heptagon and a pentagon, a configuration we call a "*7-5 pair*." In two fault line directions, the 7-5 pair splits a row of hexagons in two, as in Figure 4(a,b);

this change causes an increase to the number of large polygons per row in the third direction, as in Figure 4(c). The 7-5 pair also introduces a mild bend to the pattern; a sequence of pairs can be used to construct an annulus (Figure 4(d)). The polygons in these patches are not fully regular, but they have been optimized to be as regular as possible through an algorithmic relaxation process.

The fan and half fan basic shapes require us to fill wedge-shaped regions with small tiles. Figure 5(a) shows how this growth can be accomplished through the gradual insertion of 7-5 pairs. These basic shapes will terminate in a roughly perpendicular bounding curve, which is not compatible with the fault line directions in the [3.6.3.6] tiling. However, we can use a row of alternating pentagons and heptagons, as shown in Figure 5(b), to close off the wedge.

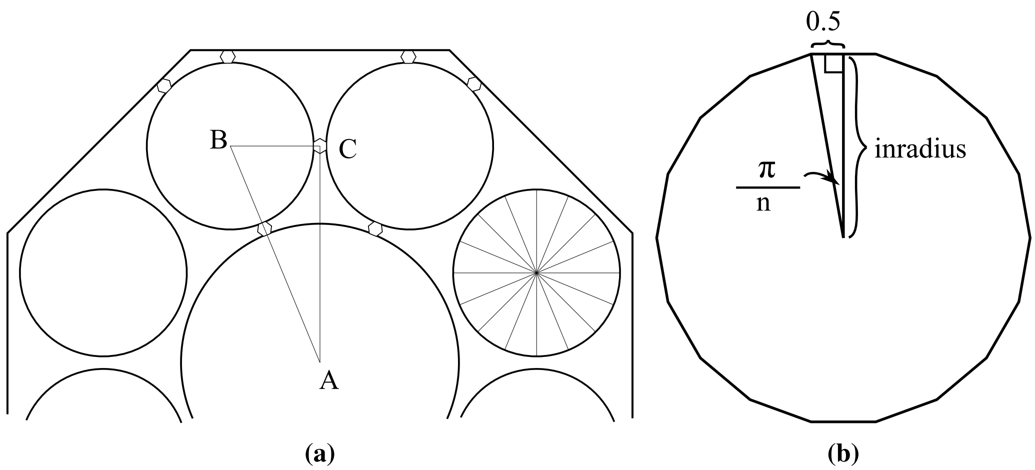


Figure 6: *(a) Spacing the rosettes; (b) inradius of a regular polygon.*

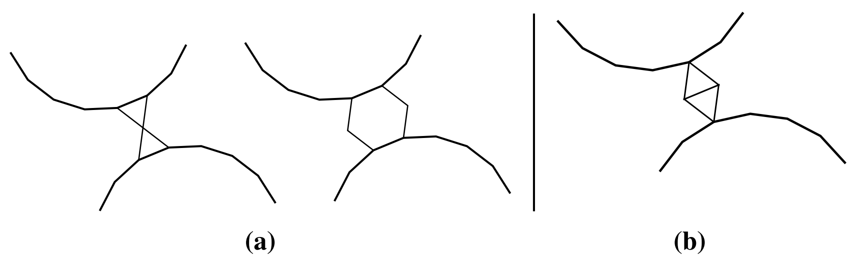


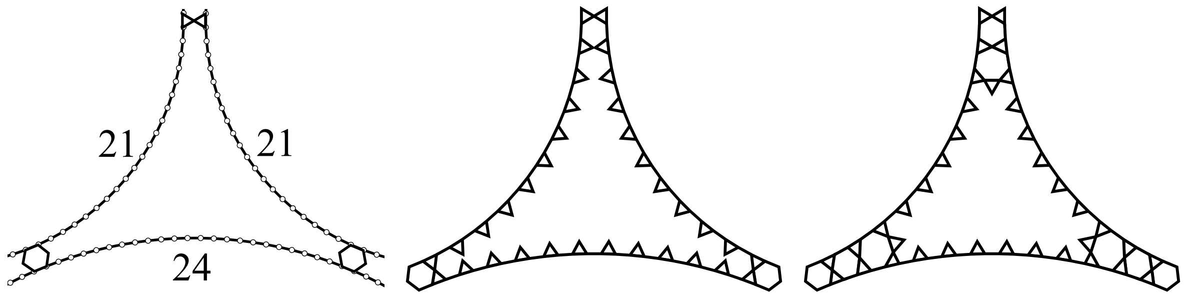
Figure 7: *Filling the space between two large regular polygons when they have numbers of sides that are even (a) or odd (b) multiples of 8.*

A 192-Pointed Rosette

In this section we use the techniques introduced above to fill basic shapes with modified [3.6.3.6] tilings, yielding an example of a complete pattern based on a central $8 \times 24 = 192$-armed rosette.

As shown in Figure 6(a), the rosette will be inscribed in a regular 192-gon centered at A. 192 is an even multiple of 8; because we will want the line AC to pass through the midpoint of an edge of the 192-gon, it follows that line AB must pass through another such midpoint, and hence that we will want the polygons at A and B to have edges facing each other. We might then separate the two polygons using one of the configurations shown in Figure 7(a) (while we would use Figure 7(b) to handle odd multiples of 8).

In this case we choose to separate the polygons at A and B by a regular hexagon; this hexagon corresponds to the cusps of the basic shapes meeting there. Let R be the inradius of polygon A as in Figure 6(b). If we assume that our regular polygons have unit-length edges, then we deduce that $R = \cot(\pi/192)/2$, which is approximately 30.555. Let r be the inradius of the regular polygon at B. Using the fact that the regular hexagon has height $\sqrt{3}$, elementary trigonometry tells us that $\sin(\pi/8) = (r + \sqrt{3}/2)/(30.555 + \sqrt{3} + r)$, from which we conclude that r is approximately 18.612. If rosette B has n arms, then $\tan(\pi/n) = 0.5/18.612$, with a closest integer solution of $n = 117$. But to construct our pattern we would like n to be an even multiple of 8, which suggests that we "round" the rosette at B and its symmetric copies to 112 arms.

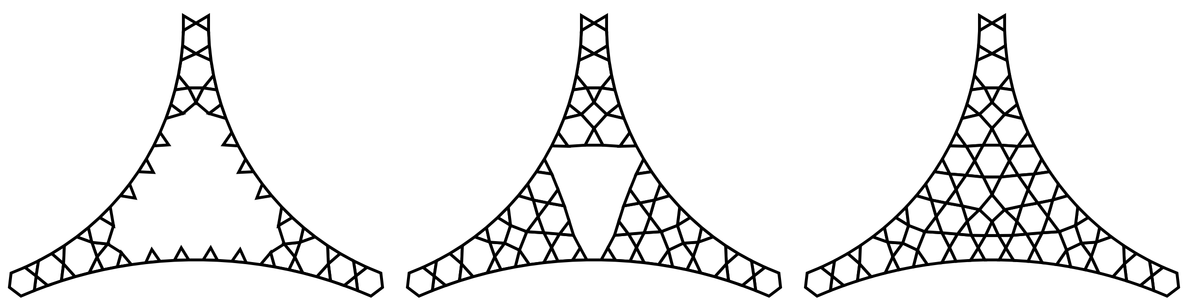


(a) (b) (c)

Figure 8: *(a) Deltoid spacing; (b) with triangles filled in; (c) filling the cusps.*

In Figure 6(a), the angle $\angle ABC$ covers 3/16 of a circle, meaning that the polygon at B will have 3/16 of 112, or 21 edges between consecutive tangencies with the polygons at A and C. Thus the interstitial deltoid will have one curve of 24 edges and two curves of 21 edges, as shown in Figure 8(a). We separate the cusps of the deltoid with two hexagons and a pair of triangles. We begin dividing the deltoid into smaller polygons by placing equilateral triangles on alternate edges, as in Figure 8(b).

We continue filling the deltoid starting at the cusps and working inward, as in Figure 8(c). We place polygons by hand using interactive drawing software written by the first author [2]. We use the distance between corresponding triangle tips to decide which polygons to use at each step. We cannot in general place regular polygons, but the software uses a relaxation process to distribute the error evenly, often producing acceptable results in which the irregularities are less noticeable.

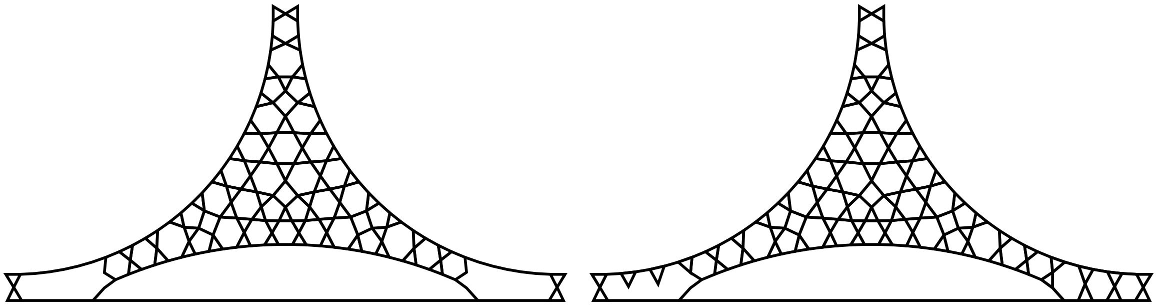


(a) (b) (c)

Figure 9: *Continuing to fill the deltoid.*

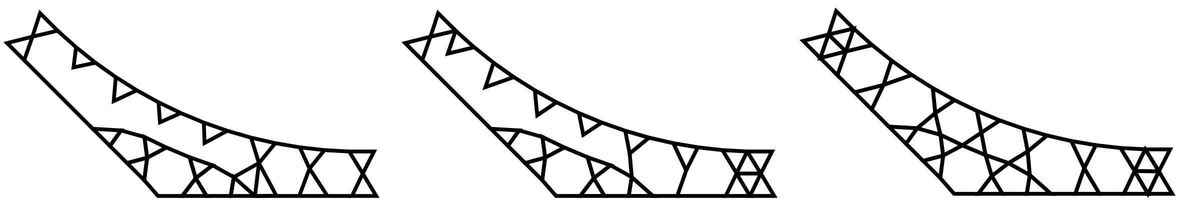
As we add polygons inward from the tips, often we can see visually what will fit next. In Figure 9(a), the gap distance has increased enough that a single polygon spanning it would be overly distorted. Instead we place a pair of pentagons. The angle between them is too large for an equilateral triangle, so we substitute a square. We continue filling in Figure 9(b). For sufficiently large deltoids, we reach a point where the unfilled space resembles a smaller, inverted deltoid, which we can then fill recursively (Figure 9(c)).

As illustrated in Figure 10(a), we can use a deltoid to fill most of an eiffel, excluding a dee and two



(a) (b)

Figure 10: *To fill an eiffel, (a) we first fill the deltoid it contains, (b) followed by the two obtuse corners in its arms.*

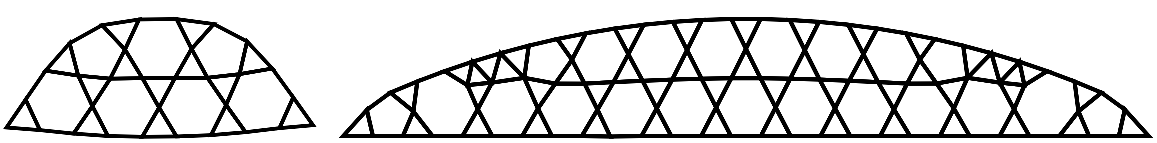


(a) (b) (c)

Figure 11: *Filling the outside obtuse corner.*

obtuse corners. We generally fill the obtuse corner with a heptagon, as in the right arm of Figure 10(b). The obtuse corner inside the eiffel is smaller than the one outside. It also has one extra curved side, but the curvature of that side is so small that we can treat it as a straight line when filling the corner. For larger rosettes, we would just have the half fan grow more rapidly after we leave the cusp of the obtuse corner. To fill the obtuse corners near the vertices of the octagonal frame we start as before, adding triangles to alternate edges. We put a heptagon in the corner, flanked by pentagons. This configuration is shown in Figure 11(a), together with an attempt to fill the area near one cusp with hexagons. Those hexagons are forced to be too stretched, which we resolve by bridging the cusps with hourglass-shaped onfigurations of triangles as shown in Figure 11(b). That configuration switches the locations of the triangles to the other set of polygon edges along the boundary of the enclosing shape, which also allows us to insert a less distorted heptagon and complete the fill in Figure 11(c).

The dee is an odd basic shape, because it has no cusps; instead, it has 60° angles at its corners. Figure 12(a) shows a small dee. The dee in our 192-armed rosette pattern (Figure 12(b)) starts very shallow and stays low. For this reason, we start with a 7-5 pair, transition to one and a half rows and finally to two rows of hexagons. We could increase its width by extending the center section more. Both Figures 11(c) and 12(b) show how we can add triangles as fillers when needed.



(a) (b)

Figure 12: *(a) A relatively small dee and (b) the dee used in our 192-armed rosette.*

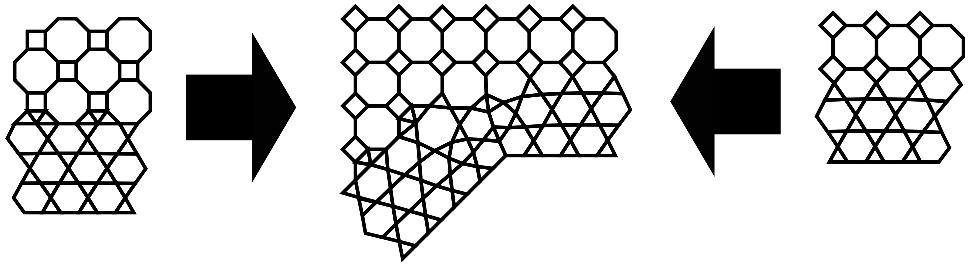


Figure 13: *Changing from [3.6.3.6] to [4.8.8] – two kinds of edges and at corner.*

Finally, we have to fit our octagonal frame into a field of smaller tiles. In a similar manner to our previous work [1], we create a transition zone that moves from a modified [3.6.3.6] tiling to the [4.8.8] Archimedean tiling by octagons and squares (Figure 13). We use two types of transition depending on whether a fault line of the [3.6.3.6] tiling is oriented parallel to an edge of a square in the [4.8.8] tiling or at an angle of 45° to it. At the corners of the large octagon frame, we have to reconcile these two relative orientations. These corners can take different forms, depending on whether the [3.6.3.6] tiling has a triangle or hexagon there, and where they meet the octagons. An example is shown shaded in grey on the left side of Figure 14, which shows the complete assembly. The basic shapes are shaded different colors, as well as both types of transition. On the right we apply the polygons-in-contact method to generate motifs for all the polygons, placing rosettes in the large regular polygons. The rest of the finished design follows by symmetry, and a complete drawing is included in the supplementary material.

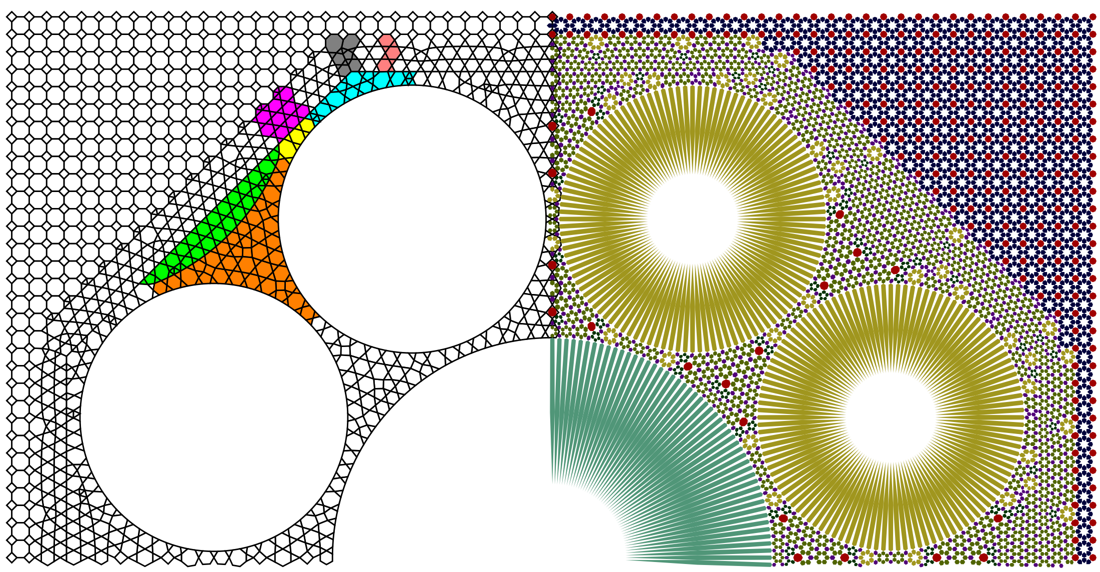


Figure 14: *The final design. We assemble the basic shapes, (a) filled with smaller polygons, and (b) generate motifs for every tile using the polygons-in-contact method.*

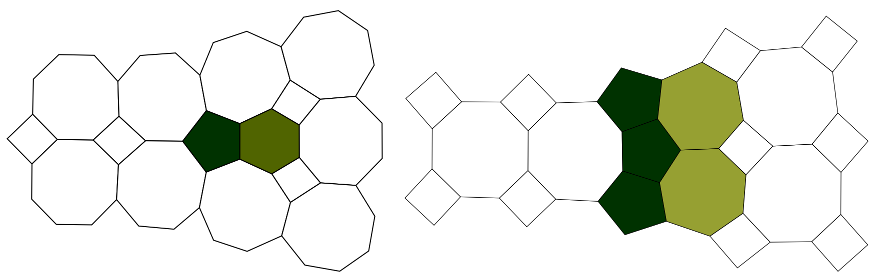
Notice that in the strip of [3.6.3.6] tiles between the octagonal frame and the surrounding [4.8.8] tiling, we have used occasional 7-5 pairs. Including these pairs makes it easier to maintain a consistent edge length across all tiles.

If we print the full design on a sheet of letter paper, the individual tiles are generally too small to see clearly. We encourage the reader to zoom in on sections of the full design included in the supplement in order to view the fine details. There are several alternate designs with different angles for the rosettes as well as some designs with stars. We also include a design with a 2024-armed rosette; in that case, it is not clear whether any printed version will be large enough to appreciate the full design.

Summary and Conclusions

As with many other decorative traditions, the development of Islamic geometric patterns is sometimes attributed to *horror vacui*, the abhorrence of an empty, undecorated surface (Gombrich preferred the more upbeat phrase *amor infiniti* [5]). Certainly these patterns satisfy our *amor infiniti*, providing a link to a legacy that still captivates us today and motivates us to seek new constructions in the same style.

The work presented in this paper fills the interstitial regions between rosettes with polygons based on a modified [3.6.3.6] tiling. Other systems are possible; for example, it would be interesting to explore methods of filling based on the [4.8.8] tiling, using devices like those shown in Figure 15 to construct wedge-shaped arrangements of tiles. More ambitiously, we might envision a general-purpose procedure that fills interstitial regions automatically. For example, it may be possible to construct a constrained Delaunay triangulation within a region [7] and extract large polygons by grouping its triangles.



(a) **(b)**

Figure 15: *Two ways to grow [4.8.8]. These are similar to the 7-5 pair for [3.6.3.6].*

Instead of using an octagonal frame, it would also be interesting to place the central rosette and its satellites inside a skeleton of eight-pointed stars and safts, as in the techniques presented by Castera for integrating large rosettes into patterns [4]. An alternative frame of that type would allow the entire composition to embed more easily within a larger field of zellij-style tiles.

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