The Rulpidon and 9-Color Maps—Supplement

Sylvie Benzoni-Gavage

Université Claude Bernard Lyon 1 and Institut Henri Poincaré, Paris, France

Abstract

We give assembling instructions for the Rulpidon pattern, along with visual explanation of why the topology of the Rulpidon is that of a three-holed torus. Additional comments are made on the two types of 9-color maps that are being considered.

Geometry

The two types of pieces of the Rulpidon pattern—shown on Figure 3 in the main paper—must be printed at the same scale and cut carefully in order for the assembling to work well, especially at the junctions of inner and outer cylinders. They can be assembled with tape hidden 'inside' the Rulpidon, starting with the inner cylinders. Middle-school and high-school students perform very well in this task. Step 1 below prepares the pattern pieces once they are cut. The actual assembling steps 2 to 6 are doable in less than 10 minutes.

Assembling instructions

1. Before assembling any pieces, put some discreet or erasable landmarks at a few remarkable junction points of inner and outer cylinders. Use for instance the landmarks shown on Figure 1 or draw your own with rule and compass on spare pieces of the pattern before transferring them on actual pieces.
2. For each of the two mask-shapes shown on Figure 2, tape together the straight lines so a to make a holed cylinder. Pay attention to which side of the paper you want to be visible at the end.
3. The two holed cylinders must be assembled along curves of the same color (yellow, orange, green, purple). It actually suffices to tape them together at the two 4-color crossing points—disregarding the color of straight lines. The result must be a cross-like shape, or bicylinder. See Figure 3 (right) where this bicylinder is shown projected on a plane orthogonal to the cylinders' axes.
4. The four pieces of outer cylinders shown on Figure 4 must be assembled along curves of the same color (yellow, orange, green, purple). Start by taping them together at either one of the two 4-color crossing points to get a sort of 4-petal flower.
5. Using your landmarks from Step 1, assemble step-by-step the bicylinder obtained in Step 3 with the holes of the flower from Step 4 along lines of the same color (cyan, blue, light pink, light red).
6. Finish the construction by taping the second 4-color crossing point of outer cylinders.

Topology

Laypeople, and even professional mathematicians when they are not used to the topology of surfaces, do not usually see at once that the Rulpidon is a genus-three surface. Rather, they tend to see four holes in it. Below is a visual explanation of why the Rulpidon has three holes, or more rigorously speaking its genus equals three.

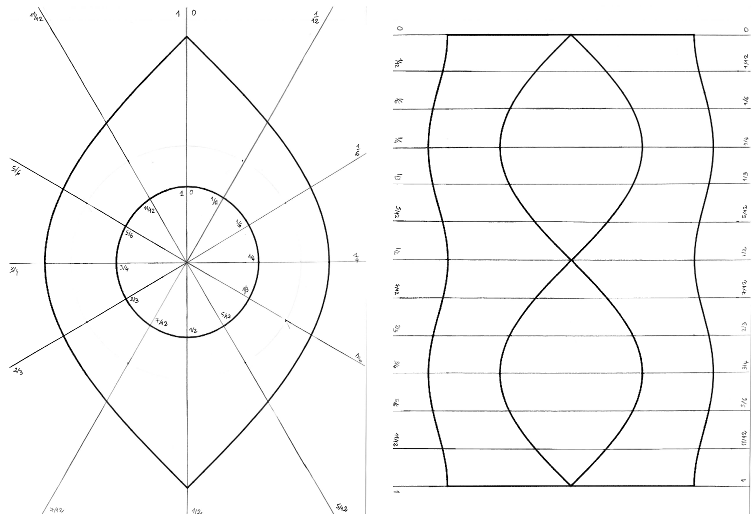

Figure 1: *Landmarks on Rulpidon pattern for junction points between inner and outer cylinders.*

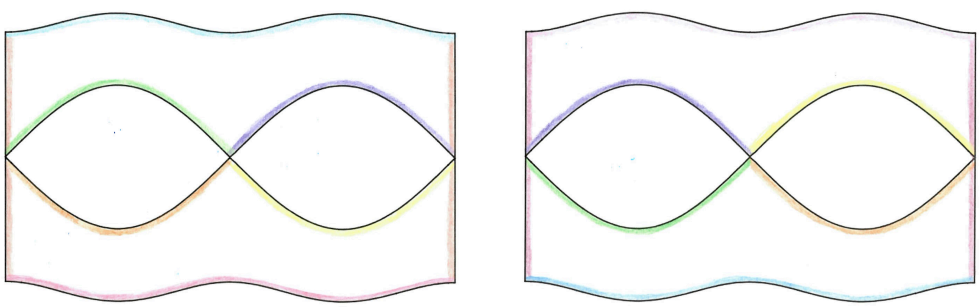

Figure 2: *Pattern of inner cylinders of a Rulpidon for which* $r = R/2$*, to be assembled along colored lines.*

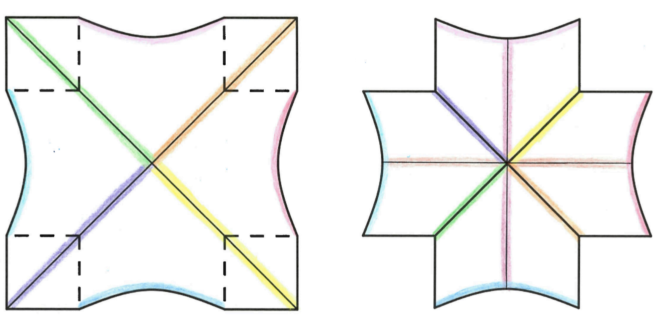

Figure 3: *Projections of assembled Rulpidon: outer surface (left) and inner surface (right).*

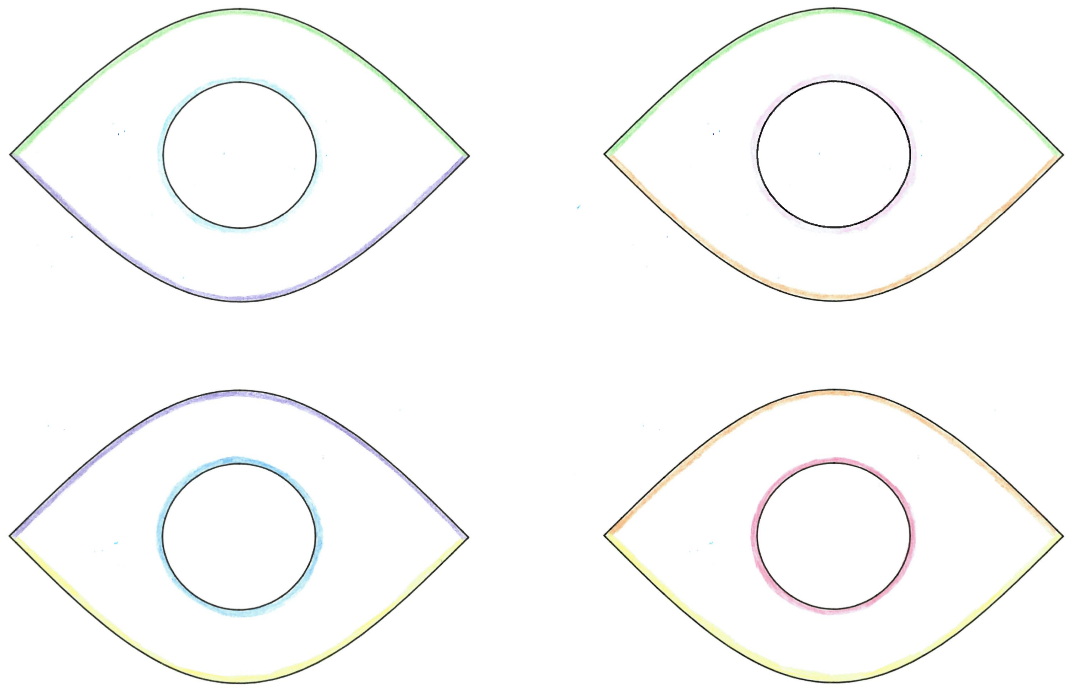

Figure 4: *Pattern of outer cylinders of a Rulpidon for which $r = R/2$, to be assembled along colored lines.*

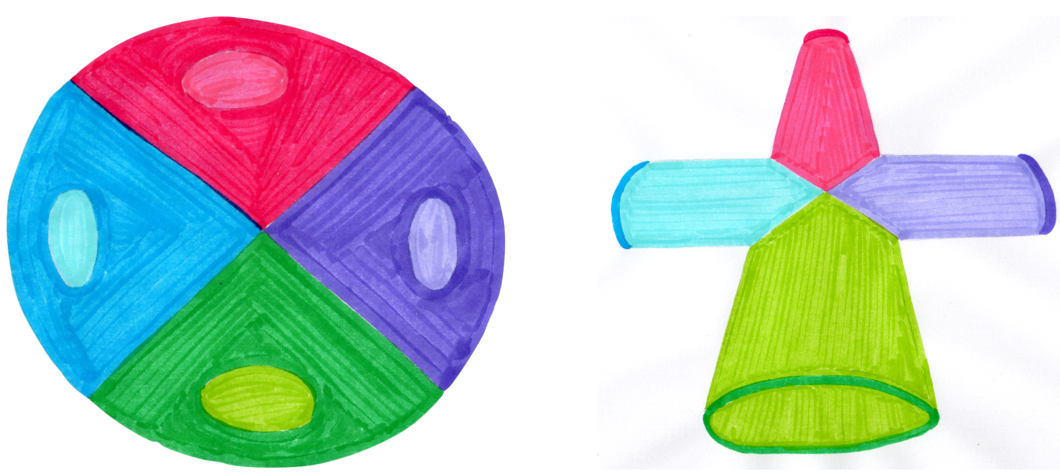

Figure 5: *Freehand drawn and colored Rulpidon: seen from above (left), inner bicylinder (right).*

Let us use colors to distinguish between the various parts of cylinders composing the Rulpidon: dark blue, green, pink and purple for the outer cylinders, light blue, green, pink and purple for the inner cylinders.

A view from above of a slightly crushed down Rulpidon is shown on Figure 5 (left). The four holes that people tend to see in the Rulpidon correspond to the four exits of inner cylinders on outer cylinders, shown as light blots on this figure. In order to count the actual number of holes, we mentally expand one of those exits, which might be easier to imagine on Figure 5 (right).

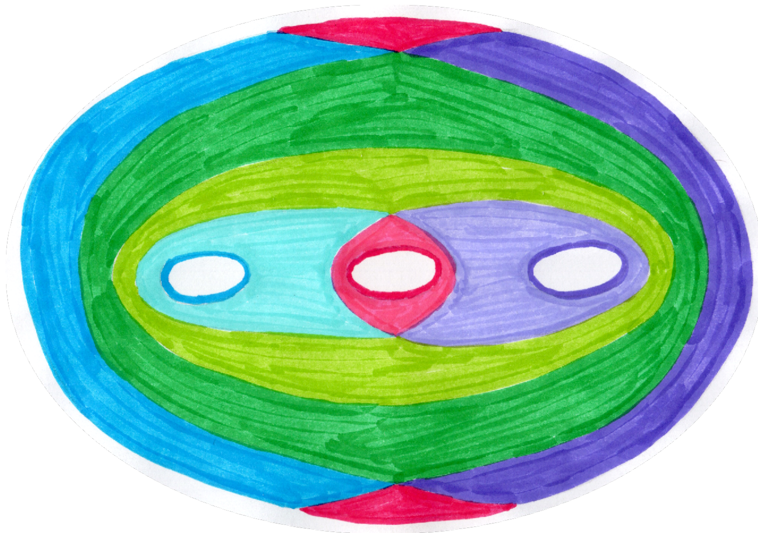

Figure 6: *Freehand drawn of crushed colored Rulpidon.*

Let us expand the green exit, and then stretch the whole solid as if it were made of clay and then crush it. The result of this mental operation is shown on Figure 6, where we do observe three holes. This means that the crushed Rulpidon, and thus also the original Rulpidon is topologically speaking a three-holed torus.

Map Coloring

The mental operation described in the previous section is a rather easy way of transferring maps between a three-holed torus and a Rulpidon. This is how I transferred Séquin's 9-map onto the Rulpidon.

Beforehand, I had designed my own map on the three-holed torus from Heffter's pattern. The way I did it induced a lot of thin, spiraling regions, which by the way I find visually appealing (see Figure 5 on main paper). I then used a different mental operation than that of Figure 6 to transfer that map onto the Rulpidon. I still find it more convenient to deal with thin regions, but it would be too long to describe here.

To finish with, I would like to point out topological differences between Séquin's 9-map and Heffter's. As I said in the main paper, Heffter's 9-map has 22 valence-three vertices—the reader might check that they can be listed as 134, 145, 156, 167, 178, 189, 192, 246, 258, 273, 284, 297, 359, 368, 376, 385, 394, 475, 487, 496, 579, 698—and one valence-six vertex (126523). In addition, all 9 regions are adjacent to each other on one and only one connected boundary line. Séquin's map, at least the one I've referred to here, has 26 valence-three vertices and not any higher valence vertex. A particularity is that 6 of its regions are adjacent to another one on two disconnected boundary lines.