Supplement to
Curved, yet Straight: Stick Hyperboloids

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STICK HYPERBOLOID TORSION ANGLE

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FOR THE CIRCULAR WAIST HYPERBOLOID OF TWO SHEETS $(b=a)$

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}\right)^2-\left(\frac{z}{c}\right)^2=1$$

ONE RULING LINE IS

$$p(u)=(x,y,z)=(a,0,0)+u(0,1,c/a) \quad \text{parametrized by } -\infty<u<\infty$$
$$=(a,u,uc/a)$$

THE DISTANCE, d, from $p(0)$ TO $p(u)$ IS $d=\sqrt{u^2+\left(\frac{uc}{a}\right)^2}=u\sqrt{1+(c/a)^2}=\frac{u}{a}\sqrt{a^2+c^2}$

SO u, AS A FUNCTION OF d IS $u=\frac{ad}{\sqrt{a^2+c^2}}$

A NORMAL AT ANY POINT (x,y,z) IS THE GRADIENT $\left(\frac{x}{a^2}, \frac{y}{a^2}, \frac{-z}{c^2}\right)$

SO A NORMAL AT $p(u)$ IS $\left(\frac{a}{a^2}, \frac{u}{a^2}, \frac{-uc}{ac^2}\right)$

OR $(ca, uc, -ua)$ BY SCALING BY ca^2

THE UNIT NORMAL AT $p(0)$ IS $(1,0,0)$

THE UNIT NORMAL AT $p(u)$ IS $\frac{(ca,uc,-ua)}{\sqrt{c^2a^2+u^2c^2+u^2a^2}}$

THE DOT PRODUCT OF THESE IS THE COSINE OF THE ANGLE ψ THAT WE SEEK

$$\psi=\text{ARCCOS}\left[(1,0,0)\bullet\frac{(ca,uc,-ua)}{\sqrt{c^2a^2+u^2(c^2+a^2)}}\right]$$

$$=\text{ARCCOS}\left[\frac{ca}{\sqrt{(ca)^2+\left(u\sqrt{a^2+c^2}\right)^2}}\right]$$

RECALL $\text{ARCCOS}\left[\frac{A}{\sqrt{A^2+B^2}}\right]=\text{ARCTAN}\left[\frac{B}{A}\right]$

$$=\text{ARCTAN}\left[\frac{u\sqrt{a^2+c^2}}{ca}\right]$$

SUBSTITUTING FORMULA FOR u FROM ABOVE

$$=\text{ARCTAN}\left[\frac{ad}{\sqrt{a^2+c^2}}\,\frac{\sqrt{a^2+c^2}}{ca}\right]$$

$$=\text{ARCTAN}\left[\frac{d}{c}\right]$$

Q.E.D.