Multilevel Islamic Geometric Design for Local Symmetry in Substitution Tilings

Jennifer E. Padilla

Altadena, California, U.S.A.; jenpadilla@gmail.com

Abstract

Substitution tilings feature a recursive generation process, lending themselves to multilevel tile design. Inspired by traditional Islamic geometric multilevel tile design, I present techniques for multilevel design based on augmented substitution rules. Islamic geometric motifs are incorporated as decoration at multiple levels of substitution to build on the recursive structure of the base-level substitution tiling. Local symmetry within two aperiodic substitution tilings is highlighted using variations of this technique.

Introduction

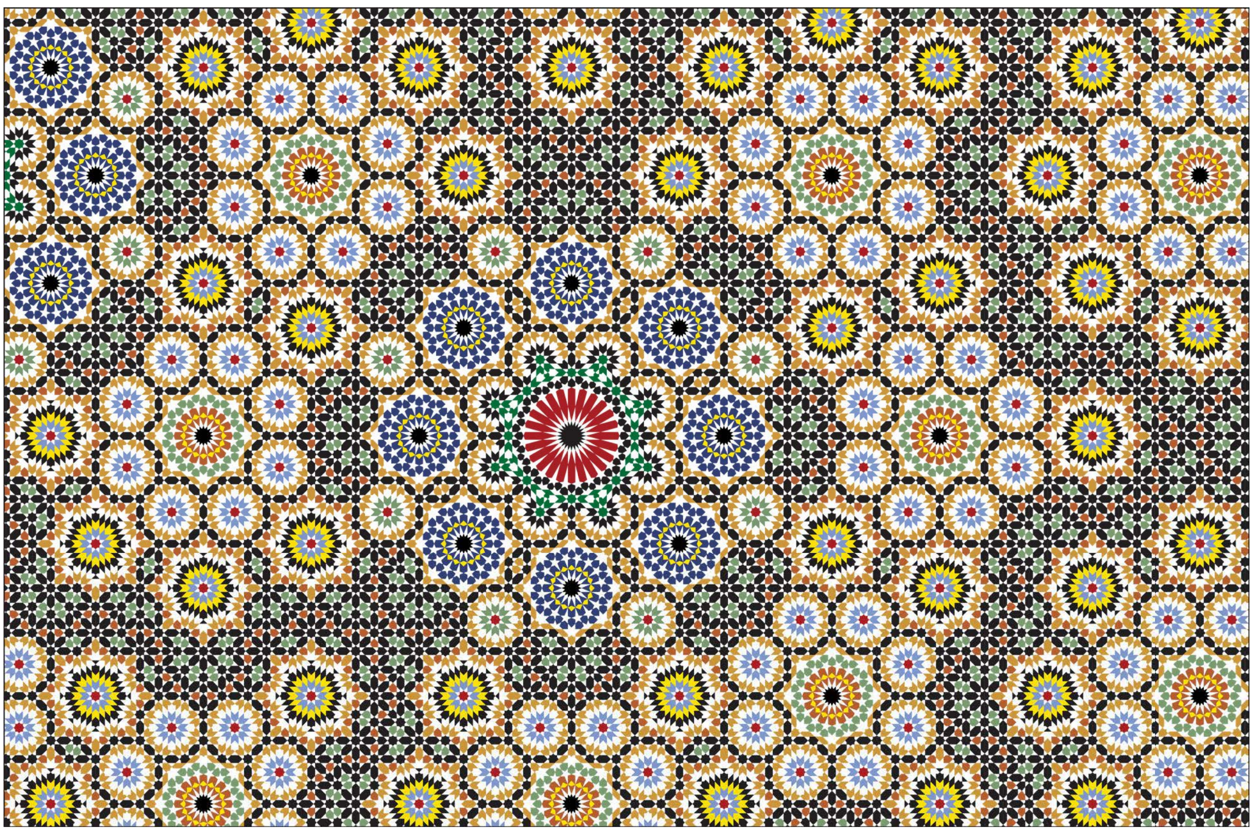

Figure 1: *The aperiodic Ammann-Beenker tiling decorated with zellij motifs. The larger rosettes are centered on local symmetry at different levels within the recursive hierarchy of this substitution tiling.*

Multilevel design with varying degrees of self-similarity is seen in many examples of traditional Islamic geometric tilings [4][16]. I apply this design concept in to the multilevel structure of two well-known substitution tilings: the Ammann-Beenker tiling and the binary tiling. Decorating substitution tilings with Islamic geometric design has been an area of rich exploration for years [1][2][5][9][11][12][13][14] in works too numerous to fully reference here. A frequently employed technique is to decorate a rhomb prototile set for a given substitution tiling with Islamic geometric motifs. Symmetric Islamic geometric motifs can be integrated into these tilings at the vertices of the rhombs as shown in [9] and demonstrated in many examples [1][5][9][11][13][14]. In this paper, the emphasis will be on expanding the decorative substitution process to higher levels in the recursive substitution hierarchy.

In Figure 1, multilevel design is achieved by decorating the Ammann-Beenker tiling, a nonperiodic substitution tiling featuring ubiquitous octagonally symmetric patches of tiles [8], with common Morrocan zellij motifs applied modularly [10] to patches of local symmetry of increasing size in the substitution hierarchy. A *patch* of tiles is defined as a finite set of tiles within a tiling whose union is a topological disk [8]. Zellij is a type of tile mosaic made from small, traditional Islamic geometric hand-cut tiles. A thorough exposition of Moroccan zellij design is given in [5], including construction of large, many-petalled rosettes that along with [3] could extend the design shown here to higher levels.

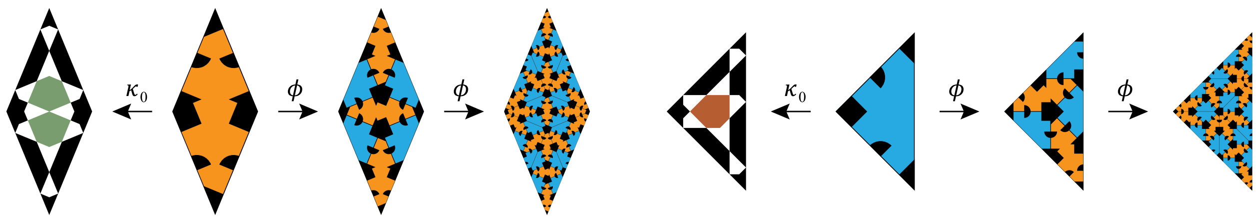

Figure 2: *Ammann-Beenker substitution rules and prototile decoration.*

Substitution tilings are equipped with an inflate-and-subdivide rule, denoted by ϕ, operating on a finite set of prototile types. The substitution rule replaces each prototile with a defined patch of prototiles. For clarity of visual presentation, it will be understood that the inflation step is not always carried out in the images. The substitution rule, with or without inflation will be denoted by ϕ. Substitution rules for the Ammann-Beenker tiling [8] are shown in Figure 2, where the rhomb (orange) and half-square (blue) prototiles are shown with two subdivision steps to their right, and a decorative substitution to the left. The term *supertile* refers to any patch of tiles derived from a prototile in the set via repeated application of the substitution rule. Prototiles are level 0, and each operation by the substitution rule, ϕ, raises the level of the supertile by one. Supertiles of any level will continue to be identified with the prototile from which they are derived. In Figure 2, the Ammann-Beenker prototiles appear with the level 1 and level 2 supertiles to their right, respectively. To the left of each prototile is the result of a decoration step, κ_0.

A substitution rule is called *perfect* if it is self-similar, that is if every supertile is similar to the prototile. Otherwise, the substitution is *imperfect* if it does not result in self-similar supertiles. More technical definitions and exposition appear in [6]. In Figure 2, the use of the half-square prototile ensures the substitution rule for the Ammann-Beenker tiling is perfect. By contrast, I use the imperfect rhomb substitution rule for the binary tiling in the second part of this paper. The presence or absence of self-similar supertiles plays a role in the choice of multilevel design methods in the two examples that follow. Another useful property shared by some substitution tilings, including the examples here, is *local isomorphism.* Another term for this property is *repetitive*. This property says that any finite portion of the tiling will be found everywhere in the tiling within some finite radius [15]. Practically speaking for design, when the tiling has this property, it means one can start anywhere with the substitution process and reach any desired finite portion after a finite number of iterations of the substitution rule. In particular, it is sufficient to generate any one of the supertiles to a high enough level to cover the area of interest.

Symmetric patches of tiles may exist within aperiodic substitution tilings where the same symmetry cannot have global scope [15]. Local symmetry is discussed with relation to the Penrose tiling in [14]. *Local point symmetry* is defined here as the point symmetry in the plane, (cyclic or dihedral), obeyed by a (necessarily finite) patch of tiles within an aperiodic 2D substitution tiling. Further, I will describe this symmetry as *geometric* if the patch of tiles obeys this symmetry only when markings on tiles are ignored, such as in the case in the Ammann-Beenker tiling whose local $d8$ symmetries apply to the local arrangement of prototile shapes (Figure 3(a)). A local point symmetry shall be called a *level N local point symmetry* if it contains complete level $N-1$ supertiles, and does not contain complete supertiles of level N or greater. Where the local symmetry has been identified and it is clear what is meant, I shall loosen the terminology and refer to it as the "N-fold," where N identifies the degree of the rotational symmetry, whether cyclic or dihedral.

The Ammann-Beenker Tiling: Zellij Decoration of Local 8-fold Symmetry

The local symmetry within the Ammann-Beenker tiling begins with the local geometric 8-fold symmetric patch in Figure 3(a). As discussed above, the markings on the Ammann-Beenker prototiles break the symmetry, but the arrangement of tile shapes has $d8$ symmetry. The recursive structure of the Ammann-Beenker tiling means the locally symmetric patch of Figure 3(a) can be found composed from supertiles at every level throughout the tiling. Figure 3(b) and 3(c) show the next two levels, comprised of level 1 and level 2 supertiles respectively, indicated visually with a semi-transparent white overlay.

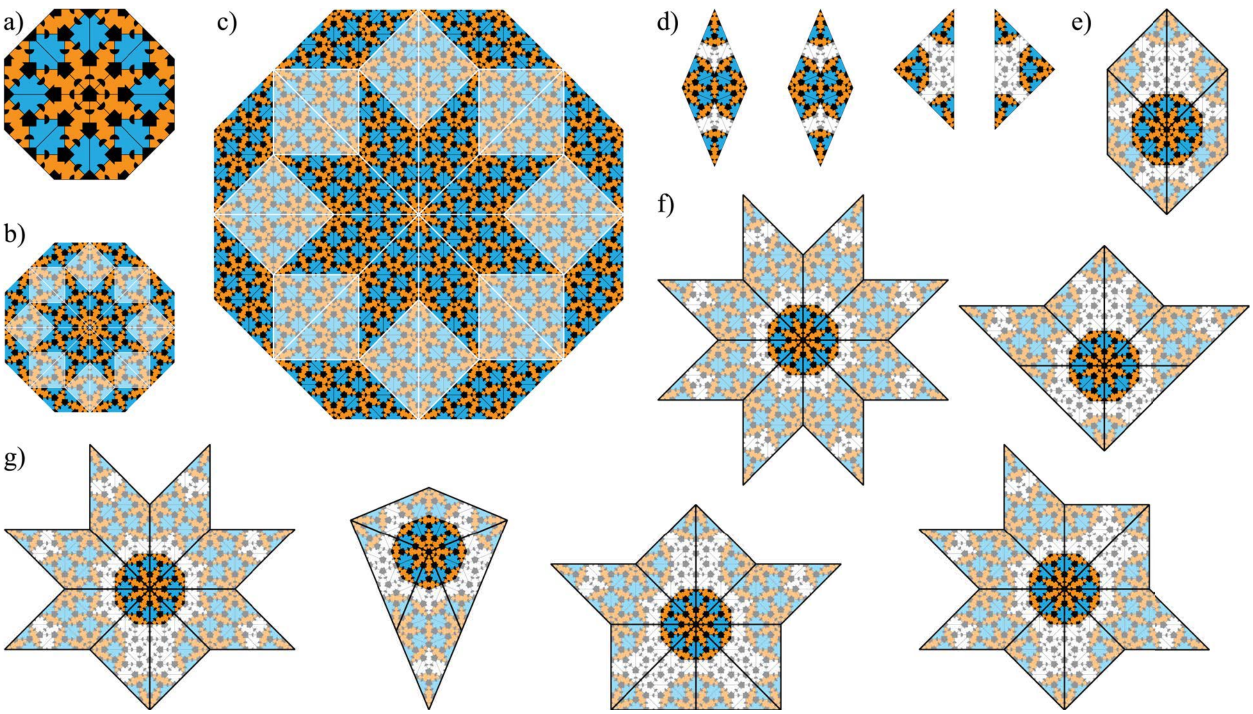

Figure 3: *Local 8-fold symmetry in the Ammann-Beenker tiling. (a) Level 1 8-fold. (b) Level 2 8-fold. (c) Level 3 8-fold. (d) Level 2 supertiles colored to show prototile participation in Level 1 8-folds. (e)-(g) Level 1 8-folds within the vertex stars of level 2 supertiles.*

The predictable appearance of 8-fold symmetric patches with respect to supertiles can be seen starting with the level 2 supertiles in Figure 3(d). These level 2 supertiles are colored to show which prototiles are involved in local level 1 8-fold symmetry. White prototiles are not involved in the level 1 8-folds of Figure 3(a). It should be noted that with the markings, the Ammann-Beenker prototile set consists of the rhomb and half square, plus their mirror images. This is true at the supertile level also. It can be seen from the arrangement of prototiles near each vertex of the level 2 supertiles, that each of these vertices will be at the center of some arrangement that is geometrically identical to the level 1 local 8-fold symmetry of Figure 3(a). *Vertex stars* are the arrangements of tiles meeting at a central vertex that are allowed by the markings of the prototiles with a given tiling. Vertex stars at the level 2 supertile level illustrate the generation of the level 1 8-fold where level 2 supertile vertices meet (Figure 3(e) – (g)). Extrapolating to higher supertile levels, level $N - 1$ 8-folds will be composed where the vertices of the level N supertiles meet. The predictable structure at each supertile level allows for supertile design by the iterative process described below.

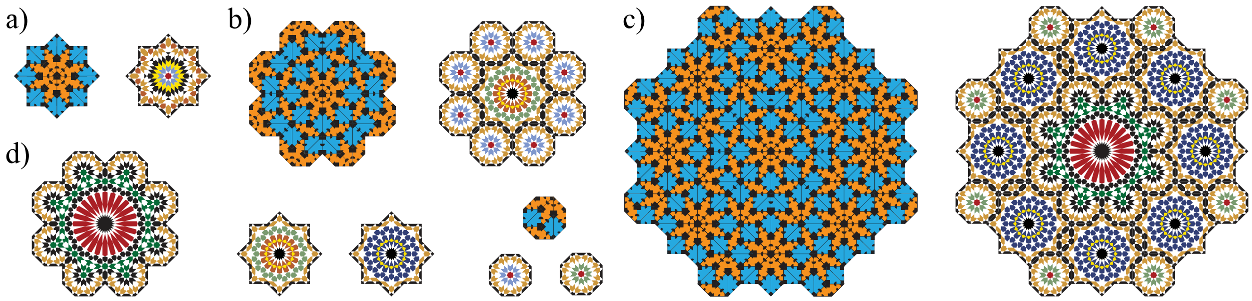

Figure 4: *Modular zellij for decoration of 8-fold local symmetry. (a) Central patch of the level 1 8-fold and the corresponding zellij decoration. (b) Central patch of the level 2 8-fold and its corresponding zellij decoration. (c) Central patch of the level 3 8-fold and its corresponding zellij decoration. (d) Modular zellij elements in the composition of the level 2 and 3 8-fold decorations.*

Modular 8-fold symmetric zellij motifs centered on rosettes of increasing number of central rosette petals are employed to indicate increasing levels of local 8-fold symmetry (Figures 1 and 4). Figure 4(a) – (c) show the central portions of the level 1 through level 3 8-folds, and how they are decorated by zelij motifs featuring central rosettes of increasing petal number. These motifs are composed modularly, including elements shown in Figure 4(d) that can replace the tile patches shown in Figure 4(a) and (b), and two modular replacements shown in Figure 4(d) below the small octagonal motif that they can replace.

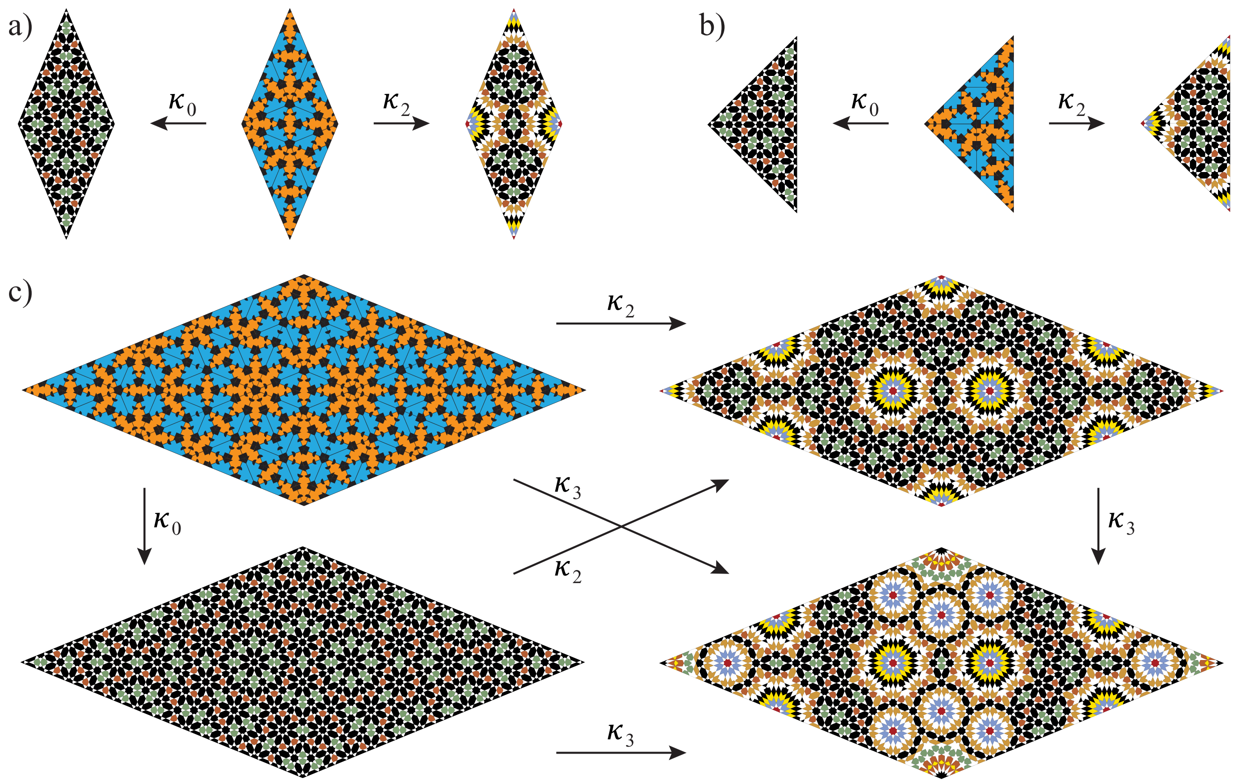

Figure 5: *Ammann-Beenker supertile decoration. (a) Decoration of the level 2 rhomb supertile. (b) Decoration of the level 2 half-square. (c) Decoration of the level 3 rhomb.*

The multilevel design process begins by adding zellij decoration at the prototile level (Figures 2 and 5), denoted by κ_0. Decoration intended specifically for the level N supertiles is denoted by κ_N. The decoration scheme here is additive in terms of the modular zellij elements, starting with prototile decoration and adding to supertiles at levels 2 through 5 (Figures 5 and 6). The decoration operation itself, κ_N, can be applied directly to any level N supertile, without going through the previous levels, once the decoration scheme is known (Figure 5(c)). The module of Figure 4(c) was recolored to serve as the decoration for the level 4 8-fold in Figure 6.

If the decorated tiles carry hidden markings from the original prototiles, this makes it possible to correctly decompose any decorated supertile into the original prototiles and their supertiles, if this was possible in the original tiling. This allows κ_N to operate on any supertile of level N or higher, by operating on the decomposition into supertiles of level N (Figure 5). The substitution operation, ϕ, can be carried out on supertiles as well as prototiles. If carried out on decorated supertiles of level N, the κ_N decoration can be simultaneously applied to all level N supertiles in the composition of the level $N + 1$ supertile.

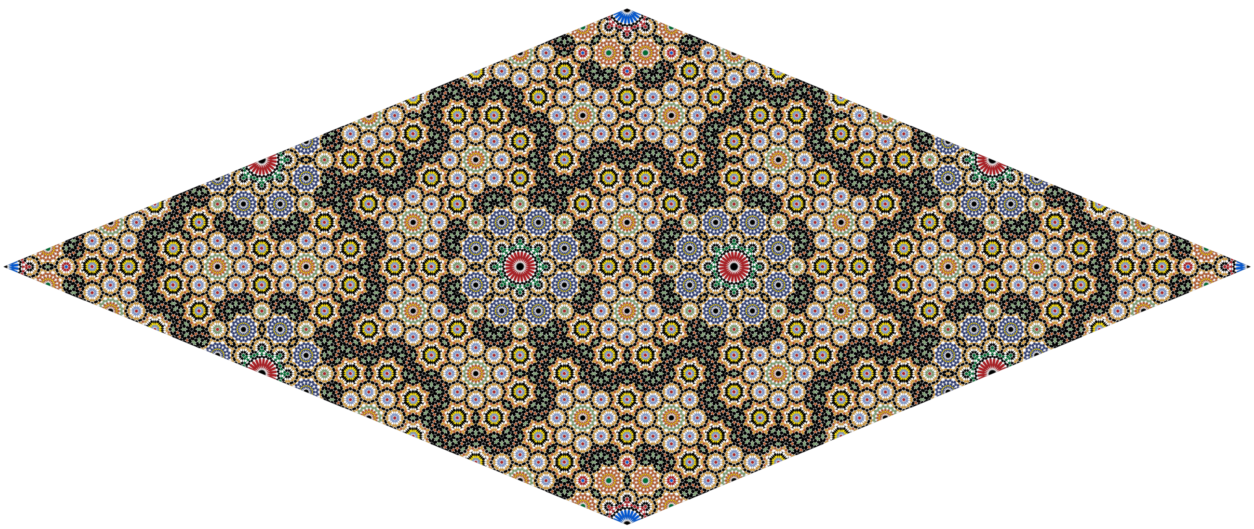

Figure 6: *Multilevel zellij decoration of the Ammann-Beeker level 5 rhomb supertile.*

Decorations can alter the local symmetry from what was present in the original tiling. The purpose here is not to represent that exact symmetry with the decoration, but rather to place an element that indicates the location of the real symmetry element, despite having extra embellishment. In this way, the underlying tiling serves as a scaffold for decoration, persisting beneath the embellishments (Figure 6).

The Binary Tiling: Local 5-fold Symmetry in Color

The binary tiling is generated by the imperfect substitution rule shown in Figure 7 [6][7][15]. It is composed of the same thick and thin rhomb prototile shapes, appearing with the same relative frequencies as in the Penrose tiling, but with different markings. The binary substitution rule has weak mixing properties that give it a higher level of disorder than the Penrose tiling [6]. Nonetheless, local 5-fold rotational symmetry is present in abundance. In this case, where supertiles are not self-similar, it is convenient to highlight local 5-fold symmetry with color, later adding Islamic geometric decoration (Figures 8 and 9).

It is not straightforward to predict the location of symmetric regions within supertiles in the binary tiling like it was for the Ammann-Beenker tiling. New patches of 5-fold symmetry arise with their centers at locations along the shared edge of supertiles when they become adjacent in the next level supertile (Figure 8). The level 5 supertiles (Figure 8(a) and (b)), and the level 6 thin rhomb supertile (Figure 8(c)) are shown with local 5-folds highlighted via color variation. Composition of these supertiles is shown with yellow lines between the supertiles at the next level down. If not already present in the composite supertiles, 5-folds appear at their boundaries within the next level supertile. The level 1 5-folds arise from the thick rhomb prototiles. It may be observed that higher-level 5-folds are composed from the thick rhomb supertiles at higher levels. Therefore, to color the 5-folds by level, it will suffice to substitute the thick rhomb supertiles for an alternate coloration according to their level when they are found to be involved in a local 5-fold symmetry. Figure 8(d) shows the use of colors from a light to dark gradient, highlighting the thick rhomb level 0 prototile in the level 1 5-fold, the level 1 thick rhomb prototile in the level 2 5-fold, and the level 2 thick rhomb prototile in the level 3 5-fold, respectively. The use of a color gradient allows easy continuation to higher levels (Figure 9(a)).

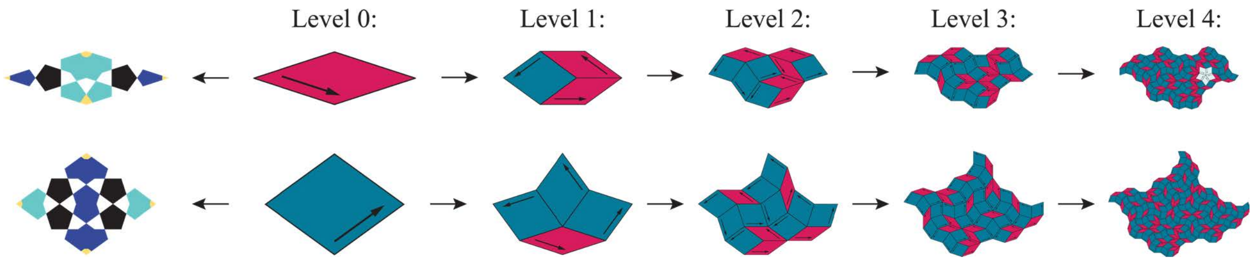

Figure 7: *Substitution (right arrows) and decoration (left arrows) of the binary tiling rhomb prototiles.*

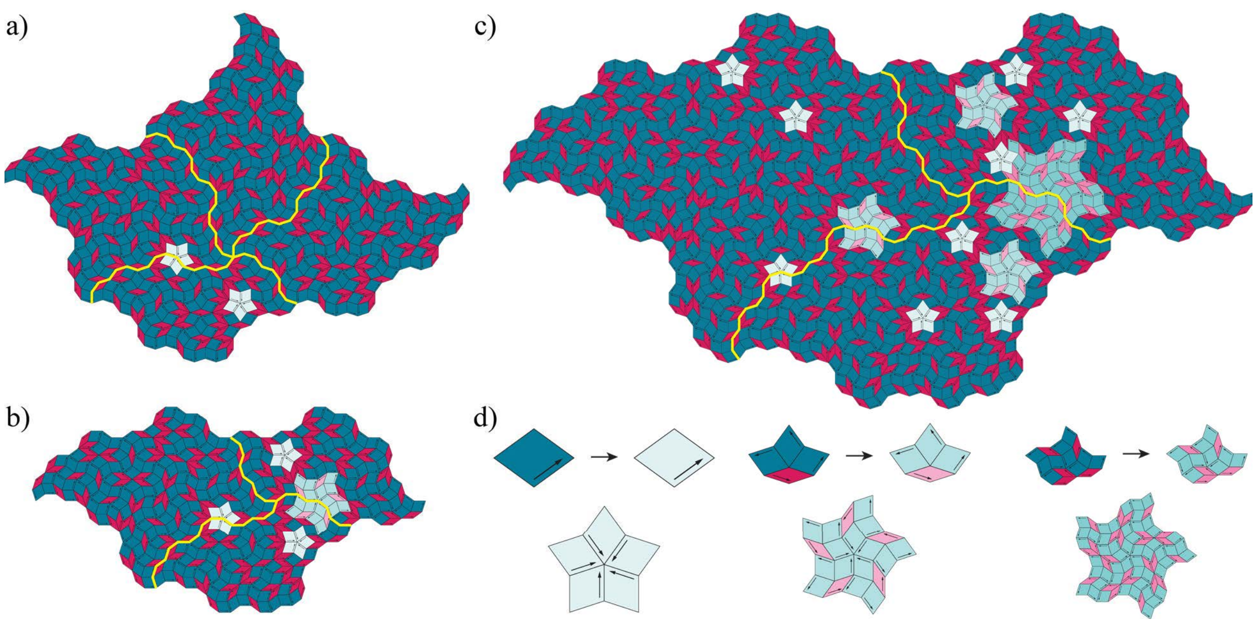

Figure 8: *5-fold symmetry arising at adjacent supertile edges.(a) Level 5 thick rhomb supertile. (b) Level 5 thin rhomb supertile. (c) Level 6 thin rhomb supertile. (d) Thick rhomb supertile coloration for local regions of 5-fold symmetry.*

As with the Ammann-Beenker tiling, the artist can choose the level of local symmetry to highlight within any given finite patch of the binary tiling. By the local isomorphism property, there will be a high enough level supertile to contain this patch entirely. Higher local symmetries always exist in the context of larger regions of the tiling. The artist may opt to highlight only up to the highest local symmetry that can be easily identified within the context of a given finite image. Since local symmetry arises centered on the edges of a supertile, the patch of interest cropped from the supertile should not be closer than r to any edge of the supertile, where r is the radius of the largest local 5-fold to be highlighted.

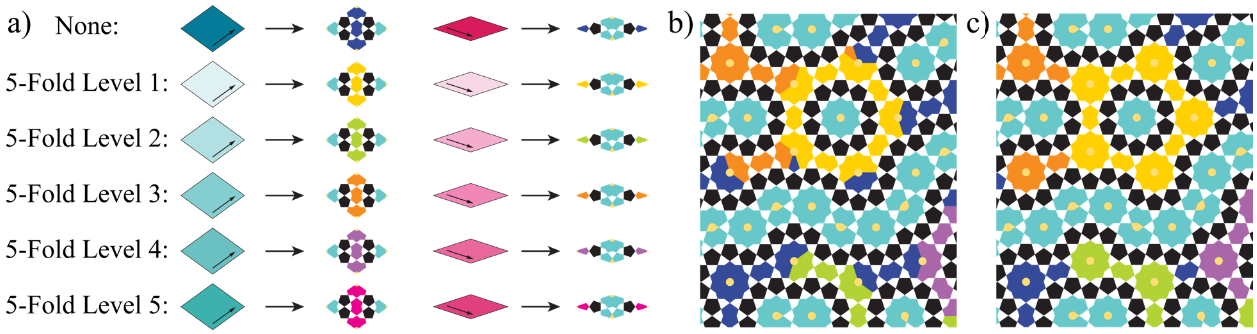

Figure 9: *Prototile decoration with coloration of 5-fold symmetry. (a) Decorative coloration scheme. (b) Direct result of prototile decoration near a level 1 5-fold. (c) Uniform coloration of decagonal motifs.*

After generating the patch of interest and highlighting local 5-fold symmetry with color, the prototiles may be decorated according to their color. Figure 9(a) shows a Nodir Devon decoration scheme [12] for the rhomb prototiles carrying the color highlights of the local 5-folds up to level 5. The rhomb prototile decoration scheme leaves an aesthetically unappealing multicolor shape within the decagonal motif of the Nodir Devon decoration (Figure 9(b)). This is easily corrected by choosing a hierarchy for the coloration scheme and replacing the entire decagonal element with a single color according to the hierarchy (Figure 9(c)). This can be thought of as an artistic intervention step, by comparison to the other techniques of this paper. Here, the hierarchy starts with the level 1 5-fold, and descends to the level 5 5-fold, with the non-symmetric coloration at the lowest level.

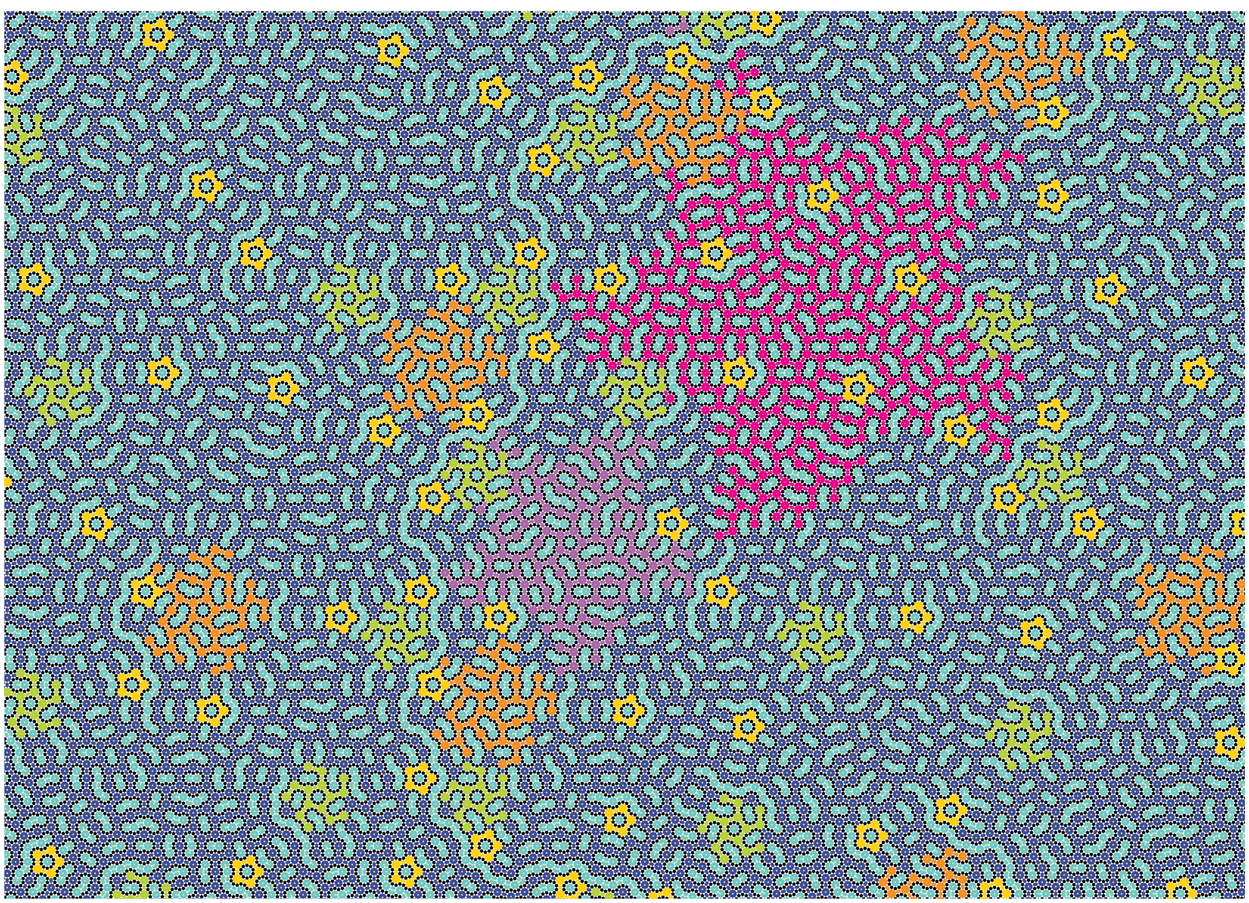

Figure 10: *Decorated binary tiling. Color highlights local 5-fold symmetry at levels 1 through 5 and the "worms" generated by thin rhombs in connected linear arrangements.*

Two main visual features appear in the binary tiling after many levels of supertile generation (Figures 8 and 10). The first is the appearance of meandering lines of thin rhombs, all in contact at their edges. I shall call these meandering lines of thin rhombs "worms." The second feature is the appearance of local 5-fold symmetries at various levels, in regular arrangements along alternating sides of the worms. In the visual disorder of the undecorated binary tiling, it is difficult to distinguish regions of local 5-fold rotational symmetry. Highlighting by color reveals them and their arrangements along the worms, creating multiscale design.

Summary and Conclusions

These two examples have explored techniques for creating multilevel design within the recursive structure of substitution tilings based on augmentation of the substitution rules. Decoration at multiple levels of supertiles visually reveals the scaling properties and recursive structure of local symmetry within these aperiodic substitution tilings.

References

[1] R. Ajlouni. "The Forbidden Symmetries." *Proceedings of the 32nd Annual Conference of the Association for Computer Aided Design in Architecture (ACADIA).* San Francisco, Oct. 18-21, 2012, pp. 391-400. https://doi.org/10.52842/conf.acadia.2012.391

[2] J. Bartholomew. *Scaling Girih Tilings*. 2010. http://joebartholomew.blogspot.com/2010/11/self-similar-girih-tiles.html

[3] J. Berglund and C. Kaplan. "Beyond the Great 96." *Bridges Conference Proceedings,* Online, Aug. 1-3, 2021, pp. 31-38. https://archive.bridgesmathart.org/2021/bridges2021-31.html

[4] J. Bonner. "Three Traditions of Self-Similarity in Fourteenth and Fifteenth Century Islamic Geometric Ornament." *Bridges Conference Proceedings,* Grenada, Spain, Jul. 23-25, 2003, pp. 1-12. https://archive.bridgesmathart.org/2003/bridges2003-1.html

[5] J. M. Casterá. *Arabesques: Decorative Art in Morocco.* Art Creation Realization, 1999.

[6] N. P. Frank. "A Primer of Substitution Tilings of the Euclidean Plane." *Expositiones* Mathematicae, vol. 26, no. 4, 2008, pp. 295–326.

[7] C. Godreche and F. Lançon. "A Simple Example of a Non-Pisot Tiling with Five-Fold Symmetry." *Journal de Physique I,* vol. 2, no. 2, 1992, pp. 207–220.

[8] B. Grünbaum and G. C. Shephard. *Tilings and Patterns*. Courier Dover Publications, 1987.

[9] C. S. Kaplan. *Computer Graphics and Geometric Ornamental Design.* Doctoral dissertation, University of Washington, 2002.

[10] C. S. Kaplan. "Generative Zellij." *Bridges Conference Proceedings,* Helsinki and Espoo, Finland, Aug. 1-5, 2022, pp. 285–288. https://archive.bridgesmathart.org/2022/bridges2022-285.html

[11] S. Moradzadeh and A. N. Ebrahimi. "3D Aperiodic Girih Tiles." *Bridges Conference Proceedings,* Online, Aug. 1-5, 2020, pp. 35-40. https://archive.bridgesmathart.org/2020/bridges2020-35.html

[12] J. E. Padilla. "Penrose Tiling Arrangements of Traditional Islamic Decagonal Motifs." *Bridges Conference Proceedings,* Helsinki and Espoo, Finland, Aug. 1-5, 2022, pp. 143–150. https://archive.bridgesmathart.org/2022/bridges2022-143.html

[13] M. Pelletier and J. Bonner. "A 7-Fold System for Creating Islamic Geometric Patterns Part 2: Contemporary Expression." *Bridges Conference Proceedings,* Towson, Maryland, USA, Jul. 25-29, 2012, pp. 149-156. https://archive.bridgesmathart.org/2012/bridges2012-149.html

[14] J. Rigby. "Creating Penrose-type Islamic Interlacing Patterns." *Bridges Conference Proceedings,* London, UK, Aug. 4-9, 2006, pp. 41-48. https://archive.bridgesmathart.org/2006/bridges2006-41.html

[15] M. Senechal. *Quasicrystals and Geometry*. Cambridge University Press, 1995.

[16] P. Webster. "Fractal Islamic Geometric Patterns Based on Arrangements of $\{n/2\}$ Stars." *Bridges Conference Proceedings,* Enschede, the Netherlands, Jul. 27-31, 2013, pp. 87-94. https://archive.bridgesmathart.org/2013/bridges2013-87.html