A Truchet Tiling Hidden in Ammann-Beenker Tiling

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Abstract

Providing an integer-only 4D inflation/deflation rule and a genuine $[x : y]$ integer projection for Ammann-Beenker tiling leads to a compelling Truchet tile pattern when setting the projection to the $[1 : 0]$ limit case. With the help of visual and syntactic equivalences with recurrences of the exact same pattern found in literature, the resulting pattern is identified and its mathematical expression explored.

Introduction

Ammann-Beenker tiling is a well-known aperiodic tiling, largely studied in the literature [2]. It is made of two tiles: a square and a rhombus of 45° angle. The Ammann-Beenker tiling is edge-oriented, and only squares needed additional information to be oriented without doubt [1]. We choose to use Truchet tiles [10, 12] to implement this orientation and adapt the tiling to provide a Truchet tiling somehow *hidden* in the Ammann-Beenker tiling.

To reach that objective, our first goal will be to provide an integer-only Ammann-Beenker tiling with elements of an integer-only 4D inflation/deflation rule in [3] and the non-overlapping rule set given in [11]. We genuinely projected the resulting tiling in a 2D integer space with two integer parameters: $[x : y]$. These parameters can provide a variety of tilings, as integer deformations from the irrational Ammann-Beenker tiling. On the one hand, by choosing a ratio (x/y) close to $\sqrt{2}$, such as $[41 : 29]$, the resulting tiling becomes indistinguishable from the irrational one. On the other hand, when the associated quotient (x/y) diverges to infinity, it leads to the $[1 : 0]$ limit case Truchet tiling, the subject of this article.

The first part will detail how we manage to provide an integer-only Ammann-Beenker tiling builder and show how this engine can provide tilings as close as we need from the irrational ideal tiling. The second part will show a peculiar set of integer parameters which provides the $[1 : 0]$ limit case Truchet tiling. The third part will compare in a visual manner the resulting tiling to existing square tilings found in the literature [4, 9]. The last part will propose a tool to compare these tilings. We will reveal compelling connections between these different tilings and formulate some new conjectures.

An Integer-Only Ammann-Beenker Tiling

Our starting point is the silver ratio $(1 + \sqrt{2})$ inflation/deflation rule proposed by F. P. M. Beenker in [1] and reproduced *with edge size indications* in Figure 1. Inspection of edge orientations in Figure 1 gives:

- on rhombuses, edges are always going from the large angle to the small angle.
- on squares, edges fork from a vertex and join to the opposite one.

And, the inflation/deflation rule is 180° symmetric on rhombuses but not on squares. To gather all the needed information on edge orientation we only need to spot each square corner *where oriented edges are joining*. In Figure 3(a), and in all Ammann-Beenker tilings displayed in this paper, we choose to indicate the distinguished vertex on each square with a Truchet tile as a square dipped in ink on the desired corner: *where*

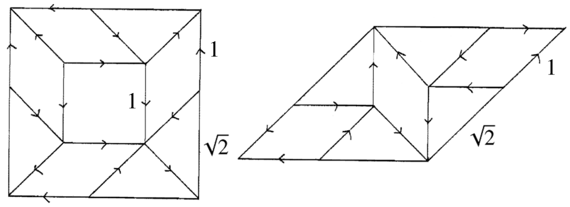


Figure 1: *Inflation/deflation rule with oriented edges, phantom half-squares and edge size indication.*

square edges join. The tile originally studied by Sébastien Truchet (1657-1729) is split along the diagonal into two triangles of contrasting colors [12]: . For example, the deflated 1×1 square shall be colored while the $(1 + \sqrt{2}) \times (1 + \sqrt{2})$ inflated square shall be colored .

As Truchet tiling involves integers, our first goal is to build an integer-only engine to obtain the Ammann-Beenker tiling. Such integer computing exists in a 4D integer lattice, as in [3], and we mix it with the non-overlapping version from [11] to avoid unneeded computing. We can then validate the correctness of the result by detecting holes or overlaps with transparency. Figure 2 presents our way to encode the inflation/deflation rule in integer-only 4D where you may observe that we have reproduced Beenker's original drawing on every tile to give two initial states that we commonly use in our software. The inflation/deflation rule can be

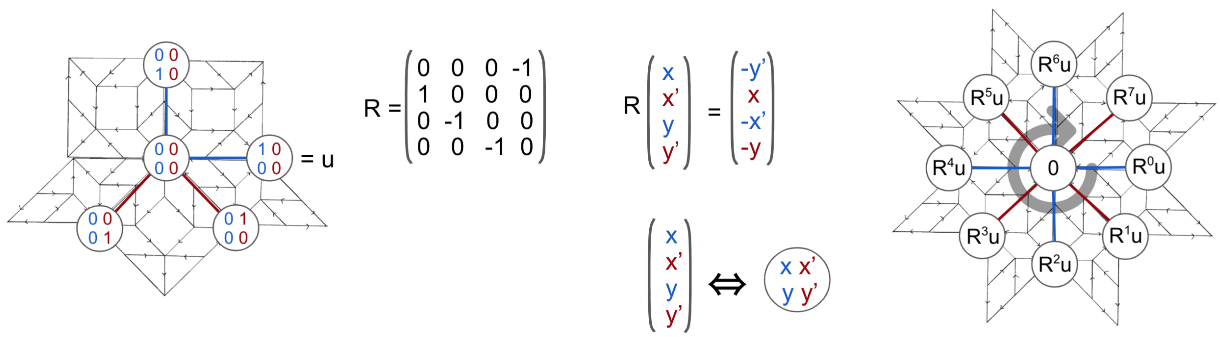


Figure 2: *4D integer lattice eight-fold rotation.*

translated using the same 4D translation. By iterating this inflation rule, we build a 4D integer engine able to produce integer-only Ammann-Beenker tiling. But the capacity to have integer-only computation in 4D would vanish if we use standard projection for 2D presented in Table 1 as it contains an irrational number "$\sqrt{2}/2$". Therefore our novel idea in that context is a projection parameterized with an integer pair denoted by $[x : y]$ as defined in Table 1. The right column shows our parameterized version where the irrational "$\sqrt{2}/2$" becomes an integer "y" parameter and the unit "1" becomes an integer "x" parameter. For any given pair of positive integers $[x : y]$, the corresponding tiling is a deformation of Ammann-Beenker tiling. Exploring this

Table 1: *Standard 45° projection and parametric version.*

45^o		$[x : y]$	
1	0	x	0
$\sqrt{2}/2$	$-\sqrt{2}/2$	y	$-y$
0	1	0	x
$-\sqrt{2}/2$	$-\sqrt{2}/2$	$-y$	$-y$

entire space is quite interesting. First, we consider the ratio (x/y). The limit case when (x/y) is diverging to infinity is the main subject of this paper and will be detailed in the next sections. At first, we choose integer values for $[x : y]$ to be as close as we want to the Ammann-Beenker tiling with an indicator established with Pythagorean theorem to verify the quality of rhombuses:

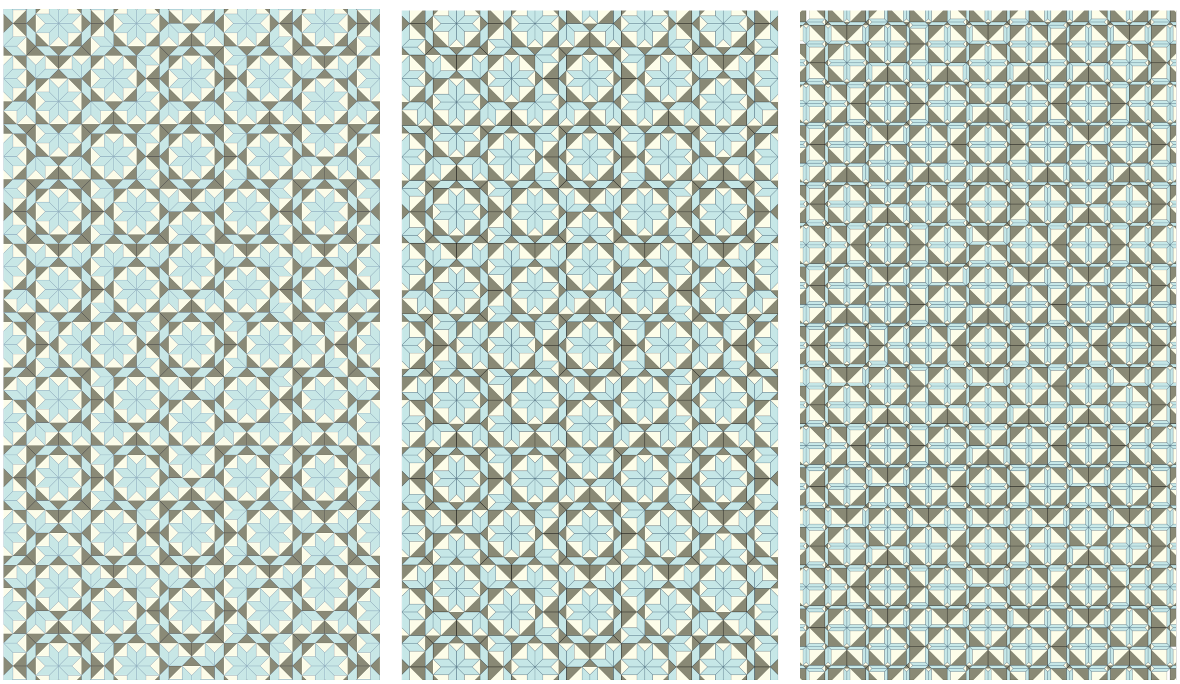
$$\Delta\,[x : y] = \frac{x - \sqrt{y^2 + y^2}}{x} = 1 - \frac{\sqrt{2}}{(x/y)}$$

From this $\Delta\,[x : y]$ we can state that the more (x/y) is close to $\sqrt{2}$, the more $[x : y]$ will be close to the ideal tiling. To reach that irrational ratio with our integer-only engine, we use the Pell sequence $P(n+2) = 2 \times P(n+1) + P(n)$, with $P(0) = 0$ and $P(1) = 1$. The sequence $[P(n+1) + P(n) : P(n+1)]$ have a (x/y) ratio that converges rapidly to $\sqrt{2}$: $[3 : 2]$, $[7 : 5]$, $[19 : 12]$, $[41 : 29]$...

Figure 3(a) is the $[41 : 29]$ tiling which cannot be distinguished from its irrational ideal as $\Delta\,[41 : 29] = 0.03\%$ - *the error is less than the image definition*. So our first goal is reached: an integer-only computing of Ammann-Beenker indistinguishable from its irrational ideal reference.

A Compelling Truchet Tiling as Integer Limit-Case

This part will extract from Ammann-Beenker tiling a Truchet tiling while exploring the other limit case when $\Delta\,[x : y]$ goes to 1. We denote this limit case by "[1 : 0]."

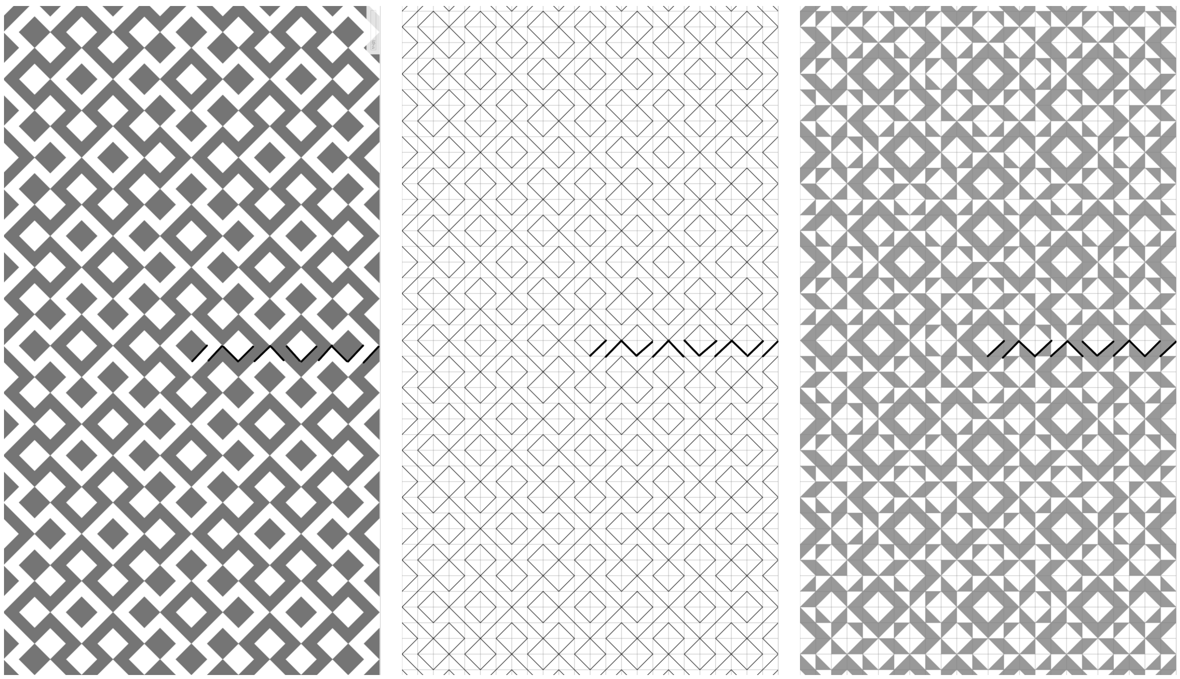


(a) (b) (c)

Figure 3: *Towards the* $[1 : 0]$ *limit case with (a)* $[41 : 29]$*, (b)* $[40 : 20]$*, (c)* $[40 : 8]$.

Property: $[1 : 0]$ is a Truchet tiling.

Figure 3 shows tilings obtained when the (x/y) ratio takes increasing values from $\sqrt{2}$ to 5. On the left, Figure 3(a) is the $[41 : 29]$ tiling, which is our case study for being very close to the irrational ideal. In the middle, Figure 3(b) is the $[40 : 20]$ tiling where $(40/20) = 2$, which is a zoomed version of $[2 : 1]$, where the deformation is visible but the overall aspect remains. On the right, when the (x/y) ratio is 5 as in Figure 3(c) the tiling appears mainly made of Truchet tiles, the other tiles being very small. Increasing the ratio will make the other tiles smaller and reduced to segments at the $[1 : 0]$-limit case. Closer to the actual proof, that would necessitate more details on the 4D calculus, [1 : 0] is a canonical 2D projection of the 4D mesh. Any of these 4D meshes is composed of 4D unit polygons. In the case of [1 : 0], the rhombuses are reduced to segments in 2D, the 45° square is reduced to a point and only the 0° square remains a 2D unit square. As these squares are oriented *where square edges join*, and as the result remains with no hole and no superposition, the resulting [1 : 0] is indeed a Truchet tiling, given in Figure 4(c).

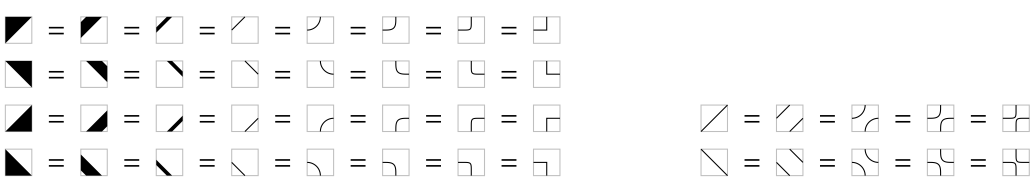


(a) (b) (c)

Figure 4: *From right to left, "Edge-to-Edge Match" coloring of* [1 : 0] *with highlighted diagonal word.*

Since the early work on aperiodic tilings, the focus was set on tile decoration implying constraints on tiles by means of edge-to-edge match. The complete description would involve deeper insights on the tile internal structure and this development can not take place here.

Definition: A Truchet tiling with "Edge-to-Edge Match" is said *linear*.



(a) (b)

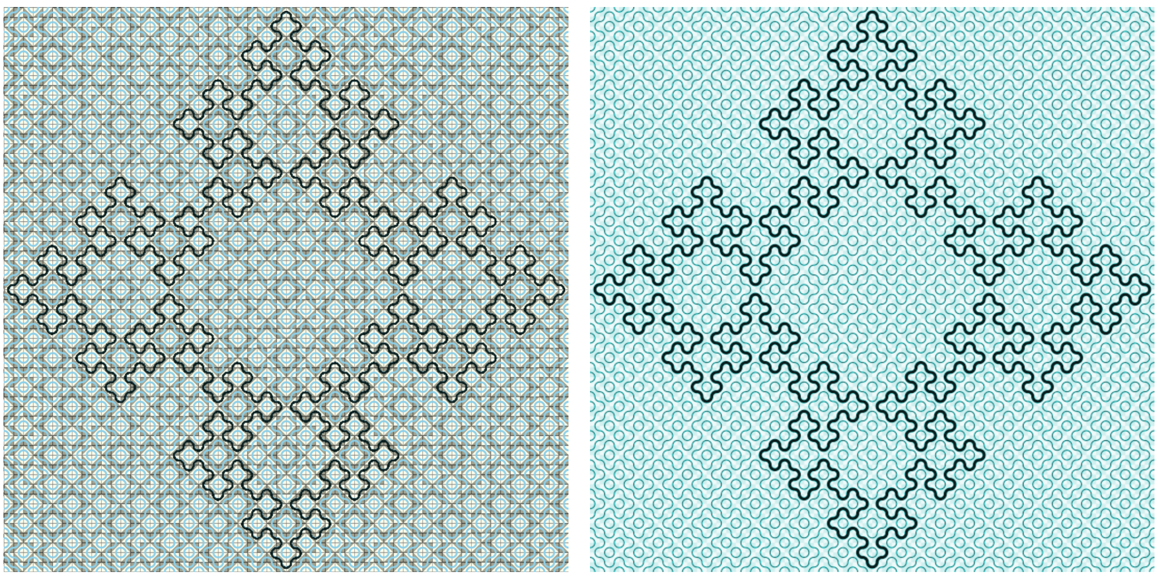
Figure 5: *Different flavors of square tiles (a) Truchet (b) Smith.*

In Figure 4 we present on left and right two Truchet tilings, Figure 4(c) is [1 : 0], and Figure 4(a) is a linear Truchet tiling. In the middle, Figure 4(b) is the common diagonal graph of both. The model and process by which we can reduce a Truchet tiling, linear or not, to its diagonal tiling is discussed more deeply in the last section, the focus at this stage being the visual comparison.

[1 : 0] as a Key to Gather Truchet Tilings Avatars

The key element of this work is to observe that the graph we found as [1 : 0] has numerous avatars in different versions of Truchet tiles and acts as a key allowing to unify these different forms. The different versions are numerous and the name Truchet tile is itself ambiguous. When rediscovered by C.S.Smith [10] in the late 80's, the current trend, linked to quasi-periodic tiling issues, was to analyze binary alphabet tiling, and so, in an homage to S. Truchet, the name "*Truchet*" was also given to a significantly different tiling with these two tiles  and . We choose to call them "*Smith tiles*," and their simplest form "*diagonal tiles*." In Figure 5, we give different equivalent forms for Truchet and Smith tiles.

In the matter of decorating Truchet tiles, imagination [6] has no clear limit, and several combinations are involved in the matches presented in Figures 6 and 7, the main argument here is the following: these four images are syntactically the same graph, and we will explain in the last part how syntactic equivalence can be extended to these kind of graphs. It is under the form reproduced in Figure 6 that we made the first

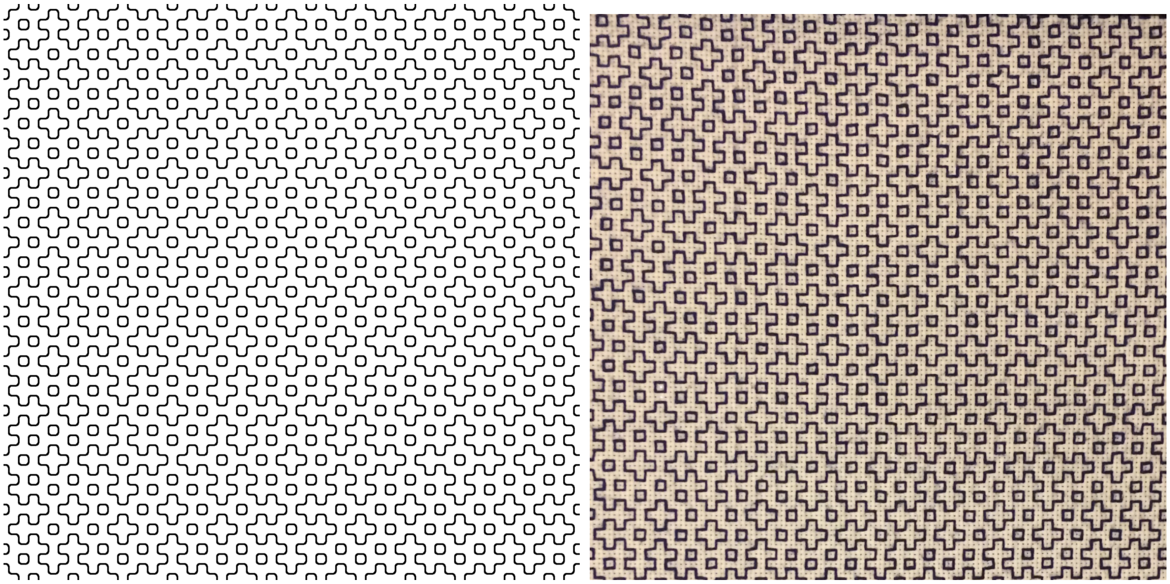


(a) (b)

Figure 6: *Match between 4*(c) *with added quarter circle decorations and Figure 2 of [4] (ArXiv version), reproduced in [5].*

match as an experimental fact which triggered that paper. The author remarks that all the "*closed curves*" formed by quarter circle concatenation share the same "*sequence of closed curves surrounding the center which develops fractal symmetry as they get larger*" [4]. The name of these shapes were identified very recently [9] in yet another equivalent version of linear Truchet tiling. In this version, a traditional Japanese stitching technique called "*hitomezashi*" is constrained to provide the same mathematical object: each thread goes left to right or top to bottom on an integer lattice and crosses the frame at each step. The closed curves spotted in [4] are identified as "*Fibonacci Flakes*" as coined in [7], but it is an experimental fact and remains a conjecture.

The remarkable observation is that both articles [9, 4] contain the *exact same graph* reproduced in Figure 6 and 7: both displays the same Fibonacci Flakes of fifth order and below. The best hint for this *exact same graph* recurrence is the main objective of [4]: defining a "*re-normalisation*" operation of which this graph is the fix point.



(a) (b)

Figure 7: *Match between an equivalent form of Figure 4*(a) *and Figure 20 of [9]*

Typographic Tilings and Resulting Crossovers

Our main tool to compare these tilings is to extend to the whole plan the (row, column) typographic coordinates: let $\mathbb{Z} \times -i\mathbb{Z}$ be the *infinite typographic plane* with infinite set of rows *and* an infinite set of columns. This definition is an evolution of the notation of [7] for the integer lattice but by changing the sign of the imaginary part, we make it compliant with standard rows and columns. To emphasize the typographic nature of the definition, we also use here the regular expression notation $[X]$ for a glyph set.

Definition: An *infinite typographic* tiling $\mathbb{T}[X] : \mathbb{Z} \times -i\mathbb{Z} \to [X]$ is a function from the infinite typographic plane to a set of glyphs $[X]$ where $(r - i.c)$ as an element of $\mathbb{Z} \times -i\mathbb{Z}$ is the coordinate of the top-left corner of the glyph at row r and column c expressed as a complex.

Syntactic equivalence: Any function $t : [X] \to [Y]$ from a glyph set $[X]$ to a glyph set $[Y]$ can be extended by composition in a function from $\mathbb{T}[X] \to \mathbb{T}[Y]$. When that function is a $[X] \to [Y]$ bijection, it extends to a $\mathbb{T}[X] \to \mathbb{T}[Y]$ bijection, providing a syntactic equivalence on an infinite tiling of square tiles.

With that simple tool and the different bijections given in Figure 5, we now try to figure out in more details the different visual equivalences provided in the previous section. On the one hand, the most simple typographic tiling is the *diagonal* tiling such as Figure 4(b) can be written $\mathbb{T}$[╱╲] : $\mathbb{Z} \times -i\mathbb{Z} \to$ [╱╲]. On the other hand, the historic Truchet tiling such as Figure 4(a) or 4(c) can be written as $\mathbb{T}$[◢◤◥◣] : $\mathbb{Z} \times -i\mathbb{Z} \to$ [◢◤◥◣]. These two tilings, even if both are called Truchet tilings in different contexts, are fundamentally different. Anyway, they can be put in relation with a non-injective function called "diagonal" which associates to each Truchet tile its color frontier between contrasting tones:

diagonal : [◢◤◥◣] $\to$ [╱╲] = (◢ $\mapsto$ ╱ | ◤ $\mapsto$ ╱ | ◥ $\mapsto$ ╲ | ◣ $\mapsto$ ╲)

In this paper, we will not give a complete algebraic definition of a linear Truchet Tiling and provide instead this elegant characterization found in [4]:

Property: For a given diagonal tiling $F \in \mathbb{T}$[╱╲] it exists a linear Truchet tiling $T \in \mathbb{T}$[◢◤◥◣] such as $F = (\text{diagonal} \circ T)$ if and only if there exists f and g such as $F(x - iy) = (f \otimes g)(x - iy) = f(x) \otimes g(y)$

with a custom multiplication in [$\diagup$ $\diagdown$] defined as:

$$\diagup \otimes \diagup = \diagdown \otimes \diagdown = \diagup$$
$$\diagup \otimes \diagdown = \diagdown \otimes \diagup = \diagdown$$

Proof hint: The proof is left to the reader in [4] but a hint is given: an third intermediate and local property is given on any unit square to allow the existence of a linear Truchet tiling for the given diagonal tiling: the product of the four values must be $\diagup$. For a rapid eye check, a negative condition is more suitable: no unit square in any 0°-90°-180°-270° rotation shall display any of the two 2×2 patterns of the multiplication table as they reduce to $\diagdown$.

But more importantly, [4] provides an example for its own purpose, a multiplicative graph $f \otimes f$, where f and g are the same, and $f(x) = (\text{true} \mapsto \diagup | \, \text{false} \mapsto \diagdown) \circ \beta$ with the reuse of the translator operator used before which means that $f(x)$ is equal to $\diagup$ when $\beta(x)$ is true, and $\diagdown$ otherwise. The function is expressed in a way the trailing $1/2$ can be used, in a percolation context, to adjust the probability of a $\diagup$ occurrence:

$$\beta(x) = \text{frac}\left(\frac{\sqrt{2}}{2} \times x - \frac{\sqrt{2}}{2} \times \frac{1}{2}\right) < \frac{1}{2}, \text{where} \, \text{frac}(x) \equiv \text{fractional part of } x$$

Conjecture: $f \otimes f = \text{diagonal} \circ [1:0]$ with $f(x) = (\text{true} \mapsto \diagup \mid \text{false} \mapsto \diagdown) \circ \beta(x)$
As provided in Figure 6 it happens this precise graph matches [1 : 0] on diagonals for any computing we could have made! These graphs show at their center Fibonacci flakes, growing like the Pell sequence $P(n)$. In [9], they provides a growing rule to compute consecutive Fibonacci flakes with the $f_{[P(n)]}$ positive initial segment of f used on rows and columns, in a similar way to the multiplicative property above:

$$f_{[P(n+2)]} = f_{[P(n+1)]} + \left(\diagdown \otimes f_{[P(n)]}\right) + f_{[P(n+1)]}$$

Each segment can be compared at each iteration with corresponding segment of the functional formula given before and this was tested over the first twenty millions terms as we did to assert our confidence in their equivalence. Here the first $f_{[P(n)]}$ Pell terms, with the diagonal translation already used and using strings as the glyph list with $f_{[0]}$ being the empty string and $f_{[1]} = \diagup$.

$$f_{[2]} = \diagup\diagup, f_{[5]} = \diagup\diagup\diagdown\diagup\diagup, f_{[12]} = f_{[5]} + \left(\diagdown \otimes f_{[2]}\right) + f_{[5]} = \diagup\diagup\diagdown\diagup\diagup\ \diagdown\diagdown\ \diagup\diagup\diagdown\diagup\diagup$$

The $f_{[12]}$ sequence is highlighted in all tilings of Figure 4.

The authors remark that the $f_{[\infty]}$ or its boolean counterpart $\beta_{[\infty]}$ sequence seemed unknown, but if we simplify the β formula of [4] by replacing the fractional part of the scalar by a modulo (_%_) operation on the floor, and provide the only offset which gives the expected match with the rewriting formula, we obtain a much simpler form:

$$\beta_{\sqrt{2}}^{-\frac{1}{2}} = \text{floor}\left(\sqrt{2} \times \left(x - \frac{1}{2}\right)\right) \%2 == 0$$

This integer sequence seems unknown precisely under this form, but it shares arbitrary long sequences with a known Beatty function with the proper glyph alphabet interpreted as integers $\lambda x : (\text{true} \mapsto 1 | \, \text{false} \mapsto 0) \circ \beta_{\sqrt{2}}^{0}$ in the On-Line Encyclopedia of Integer Sequences [8]. Neither the use of this Beatty function with a rational offset nor the use of a glyph alphabet are documented as far as we know, but the relation between Beatty functions, or their derivative or modulo, and growing patterns similar to the targeted one is widely documented in OEIS [8]. This fact leaves room for hope to connect the functional form given in [4] and the rewriting form given in [9].

Conclusion

To produce a Truchet tiling from an Ammann-Beenker tiling [1], our first goal was to provide an integer-only Ammann-Beenker tiling parameterized by an integer couple: $[x : y]$. These parameters can provide a variety of tilings as integer deformations from the irrational Ammann-Beenker tiling: either indistinguishable from the irrational one or on the contrary, a limit case displaying the the $[1 : 0]$-limit case Truchet tiling which structure is compelling.

Thanks to W. P. Hooper's work [5] exposed in 2019 Bridges Mathematical Art Galleries, we were able to recover a mathematical expression to that tiling given in [4]. In a recent work [9], a Japanese stitching technique known as "hitomezashi" is studied with an alternate formulation which identifies its loops as Fibonacci Flakes [7] and provides *exactly the same* graph! This lead to a mathematical expression for the studied Truchet tiling: an offset-ed Beatty sequence [8] of slope $\sqrt{2}$ with a modulo 2, where the only possible offset is $-1/2$. This compact form is the first goal to prove the equality between these three graphs, equality stated on an experimental basis through extensive computations.

Finally, the study of $B_\alpha^\delta \otimes B_\alpha^\delta$ where $B_\alpha^\delta = (\text{floor}\,(\alpha \times (x + \delta))\,\%2 == 0)$ and α and γ are in irrational proportions is a whole new space of exploration for those who are interested in linear Truchet Tilings [10] or their different avatars gathered here: Corner Percolations [4] or Hitomezashi Designs [9]. Whatever the name you gave to that mathematical object, there is still some mathematics to discover on their behalves.

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