The Rulpidon and 9-Color Maps

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Abstract

The Rulpidon is a metal sculpture made by the artist Ulysse Lacoste. It is an elegant genus-three surface with zero curvature; hence it can be made from sheet metal or paper. The purpose of the workshop is to draw a complete 9-color map on the Rulpidon. Complete maps are difficult to design and a complete 9-color map is not possible on a lower-genus surface.

The Rulpidon

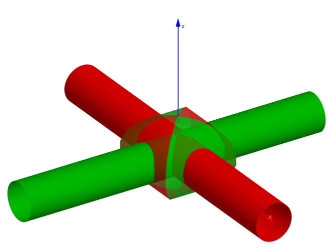
The word *Rulpidon* was invented in 2018 by the French sculptor Ulysse Lacoste to refer to a series of his works, either bronze solids of a moderate size (Figure 1(a)), or monumental surfaces made of steel sheets (Figure 1(b)). The Rulpidon is the symbol of the Maison Poincaré, a new math museum designed and hosted by the Institut Henri Poincaré. This is how I became interested in this shape.

(a) 10 *cm* ×10 *cm* ×10 *cm Bronze.*

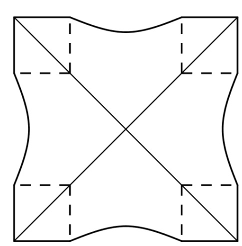
(b) 170 *cm* ×170 *cm* ×170 *cm Steel.*

Figure 1: *Rulpidon sculptures by Ulysse Lacoste [4] (photos used with permission).*

The Rulpidon geometry is rather simple. It can be thought of as a Steinmetz solid, obtained as the intersection of two cylinders of radius R at right angles, perforated by two orthogonal cylinders of radius $r < R$, see Figure 2. As a surface, the Rulpidon can be built by attaching 4 copies of the eye-shape and 2 copies of the mask-shape shown in Figure 3. The Supplement to this paper contains large versions of these shapes that can be printed out and assembled into a Rulpidon.



(a) *Perforation of a Steinmetz solid.*



(b) *Rulpidon seen from above.*

Figure 2: *Construction of a Rulpidon for which $r = R/2$.*

(a) *Eye-shape, for outer cylinders.*

(b) *Mask-shape, for inner cylinders.*

Figure 3: *Rulpidon pattern.*

From a topological point of view, the "holes" created in the Rulpidon by the inner cylinders make it more interesting than a mere Steinmetz solid. Like a three-holed torus, the Rulpidon is a genus-three surface, see Supplement. Since we can easily draw on the flat pattern, this opens interesting perspectives in terms of map coloring. I delved into this topic for a Maison Poincaré exhibit. The main purpose of this workshop, accessible to high-school students and visual art enthusiasts, is to draw complete 9-color maps on the Rulpidon.

Map Coloring

On any surface a *map* is a collection of non overlapping regions that cover the whole surface. By definition a *region* must be topologically equivalent to a disk. Two regions are called *adjacent* to each other when they share a boundary line, not just a point. A map is *complete* if all its regions are adjacent to all the others. Map coloring amounts to giving a color to each region so that any two adjacent regions have different colors. According to this rule a complete map with n regions requires n different colors and is then called a *complete n-color map*, or simply a *complete n-map*.

The 4-color theorem implies that complete maps on spheres have at most 4 colors. The existence of complete n-maps for a given n actually depends on the genus of the surface, which is zero for spheres. Genus-three surfaces do have complete 9-maps, as was shown by Lothar Heffter [3], who dealt with surfaces of genus 1 to 6. Complete 9-maps on genus-three surfaces are *maximal* in that these surfaces have no complete map with more than 9 regions. By comparison, genus-one surfaces admit complete 7-maps but no complete map with more than 7 regions. See [6] for more details. The existence proof of maximal complete maps lies in the adjacency patterns derived by Heffter. The *adjacency pattern* of a complete n-map consists of an array with n rows, one for each region indexed by k in between 1 and n. In terms of coloring, each number k can be thought of as a color. The kth row begins with "k)" and continues from left to right with the indices corresponding to the adjacent regions in the order as we encounter them on the boundary of the kth region, starting at an arbitrary point. Two of Heffter's tables are shown in Figure 4.

$n = 7, \ p_7 = 1, \ \alpha_7 = 0, \ F = 14$

regulär

1) 2 4 3 7 5 6
2) 3 5 4 1 6 7
3) 4 6 5 2 7 1
4) 5 7 6 3 1 2
5) 6 1 7 4 2 3
6) 7 2 1 5 3 4
7) 1 3 2 6 4 5

14 Dreiecke.

(a) *Complete 7-map for genus 1.*

$n = 9, \ p_9 = 3, \ \alpha_9 = 3, \ F = 23.$

1) 2 3 4 5 6 7 8 9
2) 6 1 9 7 3 5 8 4
3) 4 1 2 7 6 8 5 9
4) 5 1 3 9 6 2 8 7
5) 6 1 4 7 9 3 8 2
6) 7 1 5 2 4 9 8 3
7) 8 1 6 3 2 9 5 4
8) 9 1 7 4 2 5 3 6
9) 2 1 8 6 4 3 5 7

22 Dreiecke und das Sechseck [3 1 2 6 5 2].

(b) *Complete 9-map for genus 3.*

Figure 4: *Heffter's adjacency patterns surrounded by supplementary information from [3].*

The actual making of maps based on these patterns is not an easy task. A picture of a complete 7-map based on Heftter's adjacency pattern was given by Stahl [6]. For physical examples of maximal complete 9-maps, see the origami version by Torrence [7], and the fabric versions by Baker and Torrence [1]. For more images of maximal complete maps on other genus surfaces see the paper by Séquin [5]. Before I became aware of their work and after having unsuccessfully looked for a picture of a complete 9-map on the three holed-torus [2], I managed to draw a complete 9-map based on Heffter's pattern and then transfer it to the Rulpidon. The result is shown in Figure 5. Later, I also transferred Séquin's complete 9-map to the Rulpidon, see Figure 6.

(a) *Paper pieces of Rulpidon.*

(b) *Assembled Rulpidon.*

Figure 5: *Complete 9-color map on the Rulpidon.*

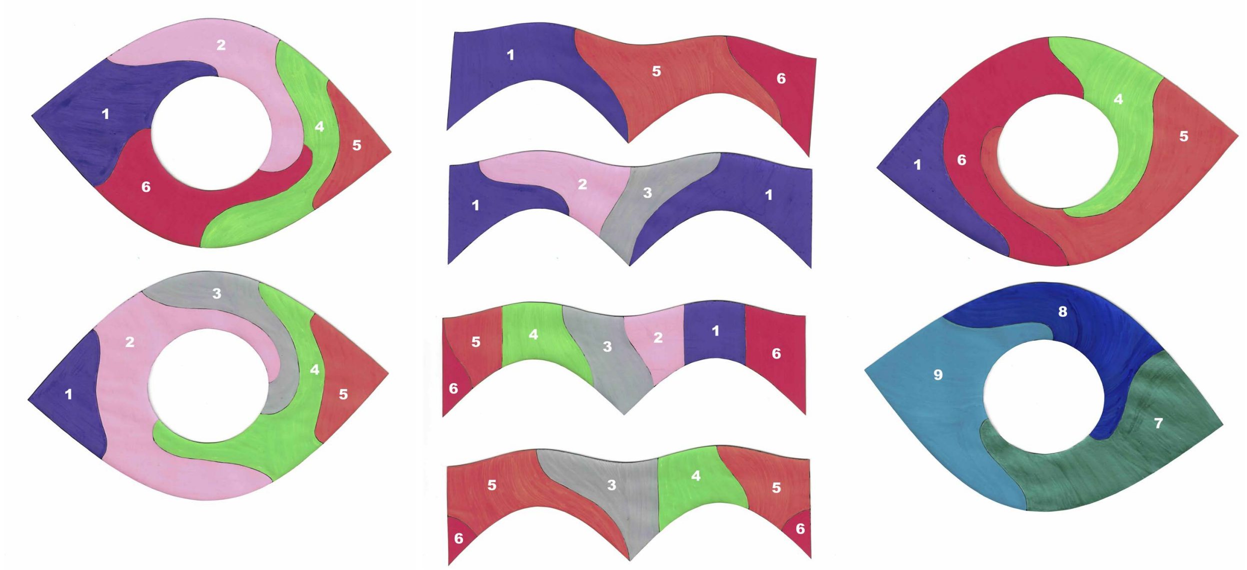
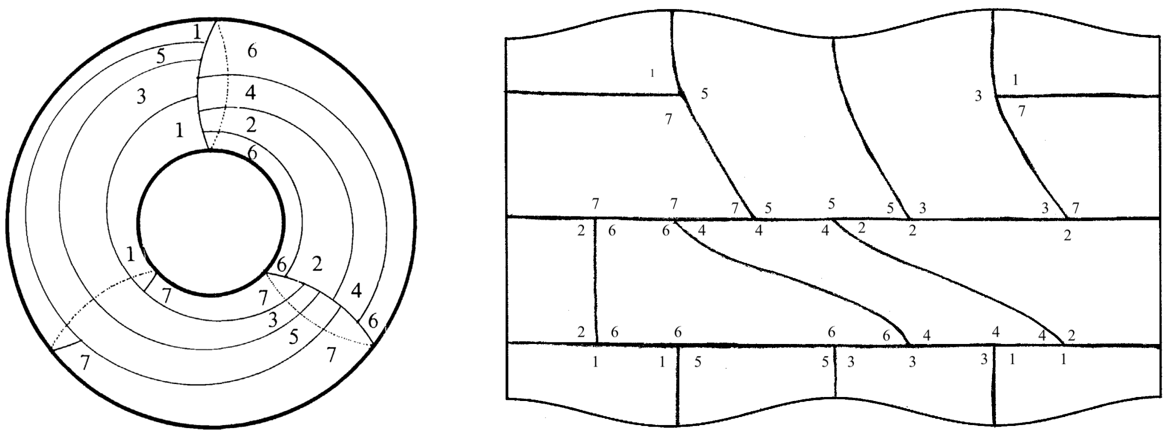


Figure 6: *Séquin's 9-color map transferred onto the Rulpidon pattern pieces.*

The goal of this workshop is for participants to learn to make their own maps. Even though my examples can serve as guides, map making leaves room for creativity. In order to get acquainted with map coloring it is important to pay attention to points where at least 3 regions meet, called *vertices*.

Heffter's complete 7-map has 14 vertices, which are all valence-3, meaning that exactly 3 regions meet at each of them. We will denote a vertex where regions i, j, k meet clockwise on Figure 7(a) by ijk, jki, or kij. Hence, we can enumerate the 14 valence-3 vertices as 126, 134, 142, 157, 165, 173, 237, 245, 253, 276, 356, 354, 452, 467, as seen in Figure 7(b). Numbering the vertices in this manner is very helpful when designing maps. Heffter calls these vertices triangles (Dreiecke). His 9-color map, described in Figure 4(b), has 22 “triangles” and one “hexagon” (Sechseck) 312652. We see that the latter has region 2 repeated. This yields a valence-6 vertex on the map where a region—the turquoise blue one on Figure 5—meets twice, which is allowed.



(a) *Heffter’s map by Stahl [6].* (b) *Flat representation of Heffter–Stahl’s map.*

Figure 7: *Heffter’s 7-color map on torus.*

Figure 7(b) shows the same (topologically speaking) complete 7-color map as in Figure 7(a). It is obtained by mentally cutting, opening up and flattening the torus. Usually, flat representations of the torus are rectangles. Here, instead of a rectangle, I have drawn the outer boundary of the mask shape from Figure 3(b).

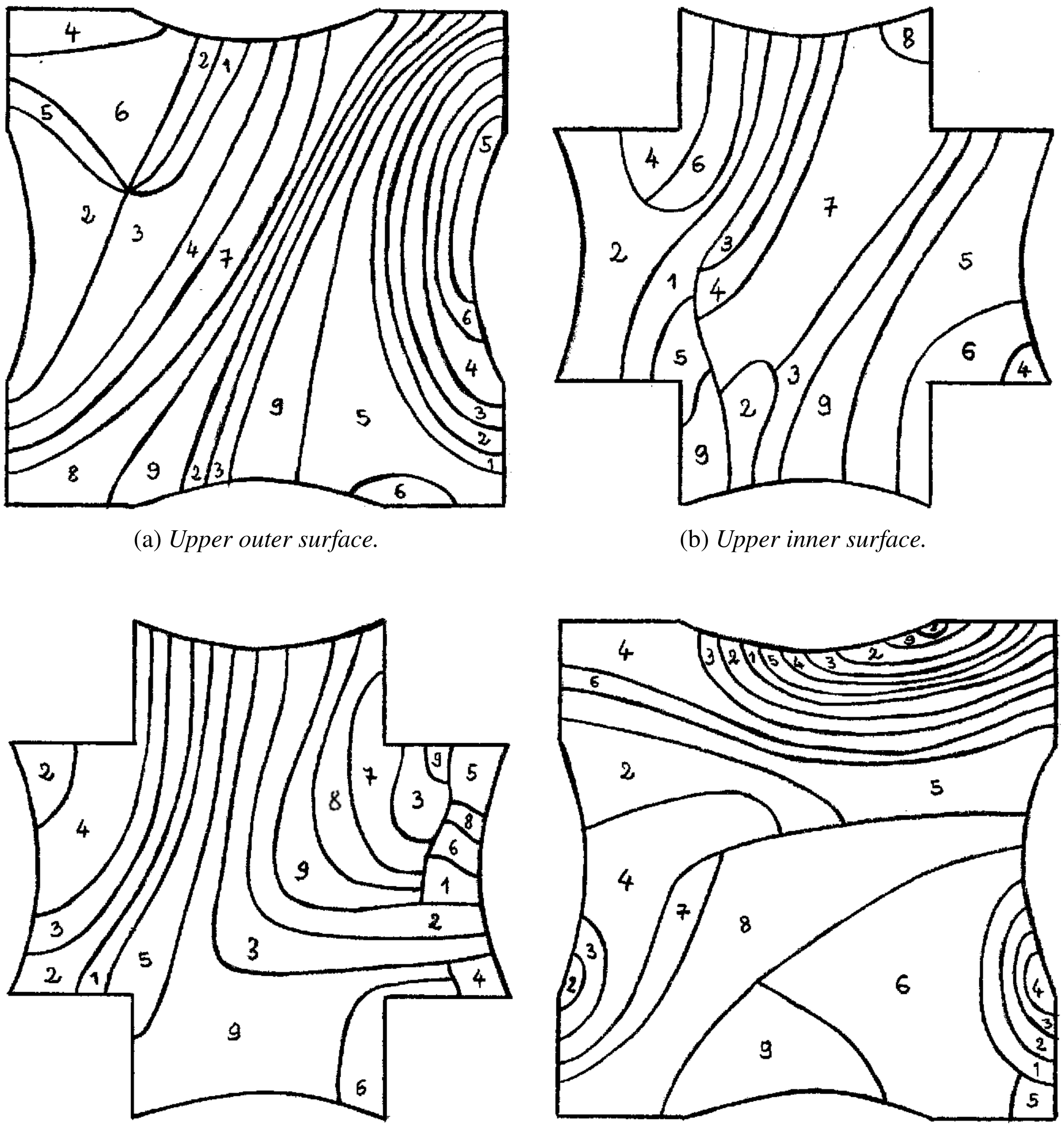
This yields an easy way of drawing Heffter’s 7-color map on a one-holed version of the Rulpidon, namely a Steinmetz solid perforated by a single cylinder. Indeed, it suffices to assemble the piece shown in Figure 7(b), which will make the single inner cylinder, with two copies of the eye-shape from Figure 3(a) and two copies of this shape without the hole, for the outer cylinders. Of course, beforehand we must continue the map onto the outer cylinders, which can be done in a most simple manner so as to connect the regions 1, 3 and 5 without introducing any new vertices. For an even more rewarding exercise, one may also try to draw other complete 7-color maps on the “one-holed Rulpidon”.

Drawing Complete 9-maps on the Genuine Rulpidon

The material needed to make a Rulpidon is 120 gsm paper printed with the pattern, scissors to cut out the shapes, and tape to assemble them. The map coloring can be made with high-quality colored pencils for best results. In addition, it is helpful to have same scale spare pieces of the pattern with landmarks to ensure that the drawings on inner cylinders continue properly on outer cylinders. These landmark pieces can be drawn with ruler and compass. The most challenging and creative part of the workshop uses pencils, erasers, and tracing paper to design maps. All the necessary material will be provided to participants.

Tracing paper is useful to visualize a whole map all at once. I used it to draw the four projections of my 9-color map shown in Figure 8, and to make sure in designing the map that regions continue properly onto all cylinders. These four pictures represent vertical projections of the Rulpidon seen from above onto a lower horizontal plane. Before projection, the Rulpidon is mentally cut along its horizontal plane of symmetry, which yields two outer, square-like pieces and two inner, cross-like pieces. As far as topology is concerned, there is no need to respect angles or distances, we can draw freehand when projecting the regions drawn on those pieces.

To begin to draw their own 9-map, participants should first find the 22 valence-3 vertices and the valence-6 vertex in Figure 8. Then they can mentally move these vertices around and make their own drawings of regions on tracing paper, as long as they keep the same collection of vertices. Once they are satisfied with the rendering, the final step is to transfer their map onto the actual pattern of the Rulpidon. This can be challenging when you are not used to it. It becomes much easier after a little training.

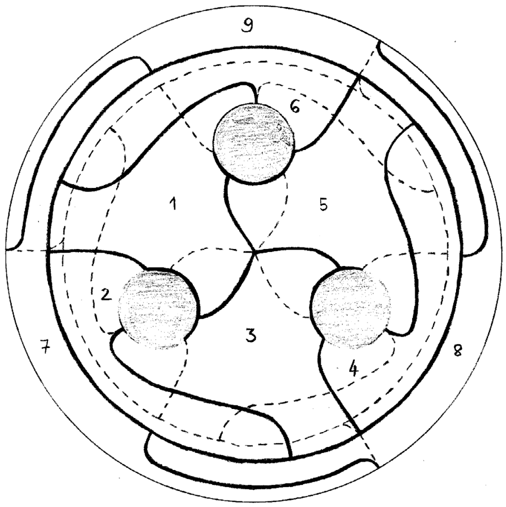


(a) *Upper outer surface.* (b) *Upper inner surface.*

(c) *Lower inner surface.* (d) *Lower outer surface.*

Figure 8: *Projections of 9-color map on the three-holed Rulpidon seen from above.*

Notice that Séquin's complete 9-map on the three holed-torus has 3-fold symmetry, see Figure 9(a) (or Figure 2 in [7]). The Rulpidon itself has 4-fold symmetry. Can we draw a complete 9-map on the Rulpidon that has 4-fold symmetry as well? The one I drew on the Rulpidon using Séquin's map does not fulfill this requirement, see Figure 9(b). This is an open challenge for the participants to attempt on their own.



(a) *Freehand drawing of Séquin's 9-color map on torus seen from above.*

(b) *Assembled Rulpidon with Séquin's 9-color map transferred onto it.*

Figure 9: *Séquin's 9-color map [5] on torus (personal drawing) and on the Rulpidon (personal work).*

Summary

I have introduced a piece of art, the Rulpidon, which is a developable genus-three surface and thus can be colored with beautiful complete 9-color maps. Participants will draw a 7-color map on a one-holed Rulpidon as a starting exercise. They will then design their own 9-color maps on a Rulpidon, using geometric pattern and map templates I have provided as an example. Finally, they will assemble their templates into a 3 dimensional Rulpidon whose surface is covered in a complete 9-map they designed themselves.

Acknowledgements

I am indebted to Eve Torrence for having answered my question on mathoverflow [2], which enabled me to meet her, Ellie Baker and Carlo Séquin. I was delighted to discuss with the three of them the making of models of maps on higher genus surfaces, and I warmly thank them for sharing their findings.

References

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